SOLUTIONS

MA 412

MA 412 – Complex Variables
Exam #1

Name:

Instructions: To receive full credit you must show all work. Explain your answers fully and clearly. You may refer to theorems/facts in the book or from class. No calculators, books or notes of any form are allowed. Good luck!

Question	Score	Out of
1		12
2		12
3		10
4		16
5		10
6		15
7		13
8		12
Total		100

1. (12 points)

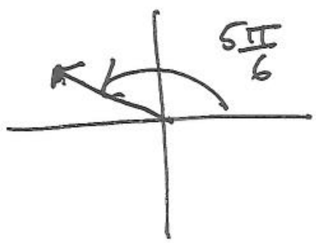
- Write

$$\frac{4}{-i-\sqrt{3}}$$

in exponential form

$$\frac{4}{-i-\sqrt{3}} = \frac{-4}{\sqrt{3}+i} = \frac{-4(\sqrt{3}-i)}{4} = -2\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)$$

$$= 2\left(-\frac{\sqrt{3}}{2}+\frac{i}{2}\right)$$



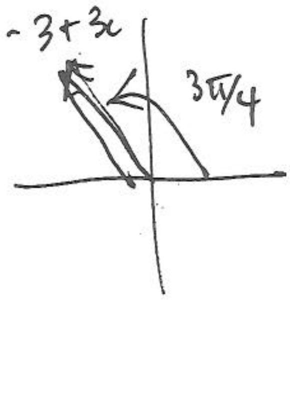
$$= 2\exp\left(\frac{5\pi i}{6}\right)$$

or any angle that differs by $\underline{2\pi k}$.

- Write

$$(-3+3i)^{27}$$

in rectangular form



$$-3+3i = \sqrt{18}\exp\left(\frac{3\pi i}{4}\right)$$

$$(-3+3i)^{27} = 18^{27/2}\exp\left(\frac{81\pi i}{4}\right)$$

$$= (18)^{13}\sqrt{18}\,\underbrace{\exp(20\pi i)}_{=1}\exp\left(\frac{\pi i}{4}\right)$$

$$= (18)^{13}\sqrt{18}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right).$$

2. **(12 points)**
Find the sixth roots of -4, i.e.

$$(-4)^{1/6}$$

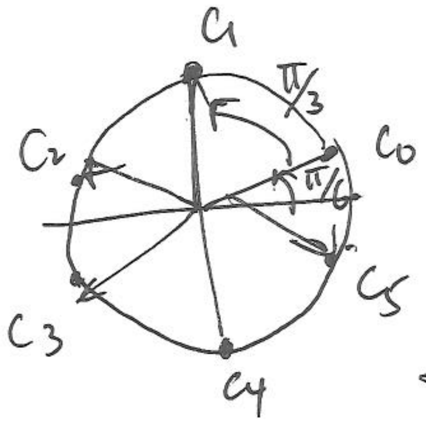
in both exponential and rectangular form, AND sketch them

$$-4 = 4\exp(\pi i)$$

$$(-4)^{1/6} = c_k = (4)^{1/6} \exp\left(\frac{\pi i}{6} + k\frac{2\pi i}{6}\right)$$

$$k = 0, 1, 2, \ldots 5.$$

$$= \sqrt[3]{2} \exp\left(\frac{\pi i}{6} + k \cdot \frac{2\pi i}{6}\right)$$



Rectangular form

$$c_0 = \sqrt[3]{2}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$$

$$c_1 = \sqrt[3]{2}\, i$$

$$c_2 = \sqrt[3]{2}\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$$

$$c_3 = \sqrt[3]{2}\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$$

$$c_4 = -\sqrt[3]{2}\, i$$

$$c_5 = \sqrt[3]{2}\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$$

3. **(10 points)**
Sketch the region described by the inequality

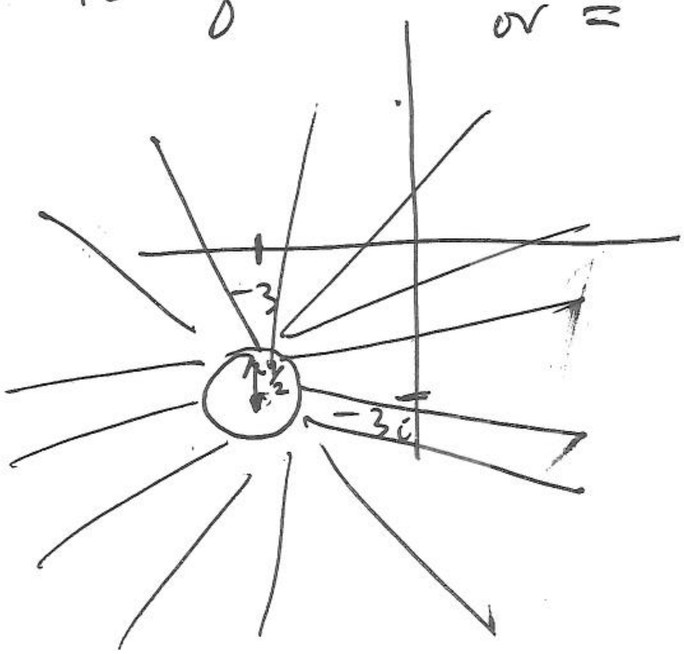
$$|2\overline{z} + 6 - 6i| \geq 1$$

$$|2\bar{z}+6-6i| = |\overline{2\bar{z}+6-6i}|$$
$$= |2z+6+6i|$$
$$= 2\,|z+3+3i| \geq 1$$

So $\quad |z+3+3i| \geq \frac{1}{2}.$

$\Rightarrow \quad |z-(-3-3i)| \geq \frac{1}{2}.$

Says the distance from z to $(-3-3i)$ is greater than or $=$ $\frac{1}{2}$.



So the region consists of all pts in the complement of the open disk of radius $\frac{1}{2}$ centred at $(-3-3i)$

4. (16 points)

- Let D be the first quadrant, i.e. $D = \{z = x + iy | x \geq 0, y \geq 0\}$. Describe algebraically and sketch the image of D under the map $f(z) = \bar{z}^2$

If $z = re^{i\theta}$

$\bar{z} = re^{-i\theta}$

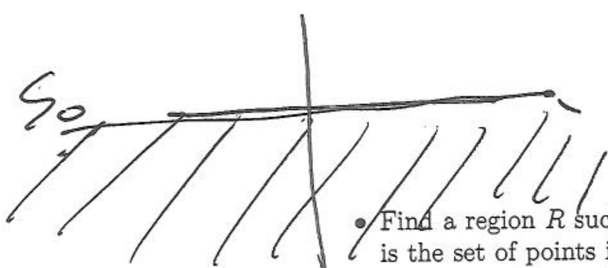
So $f(re^{i\theta}) = r^2 e^{-2i\theta}$

$D: \{(r,\theta) \mid r \geq 0,\ 0 \leq \theta \leq \frac{\pi}{2}\}$

$r^2 \geq 0 \quad -\pi \leq -2\theta \leq 0$

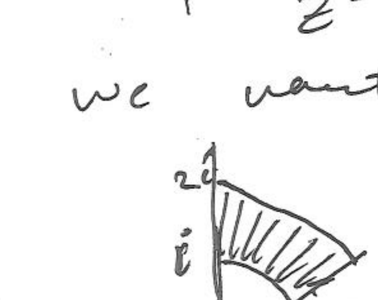
So the image is $\{z = x+iy \mid y \leq 0\}$.

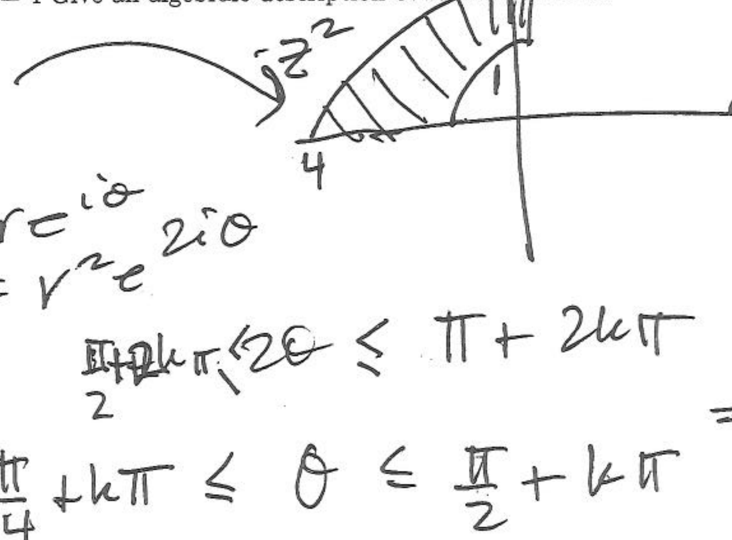
So



- Find a region R such that the image of R under the map $f(z) = z^2$ is the set of points in the second quadrant lying between the circles $|z| = 1$ and $|z| = 4$ Give an algebraic description of R and sketch R.

?





$z = re^{i\theta}$

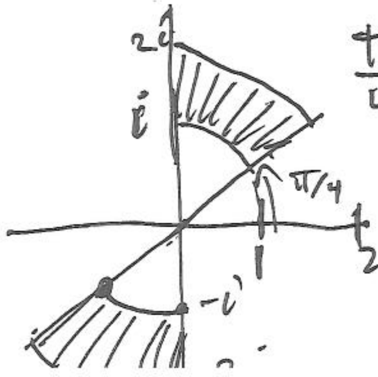
$z^2 = r^2 e^{2i\theta}$

we want $\frac{\pi}{2} + 2k\pi \leq 2\theta \leq \pi + 2k\pi \quad 1 \leq r^2 \leq 4$

$\Rightarrow 1 \leq r \leq$

$\frac{\pi}{4} + k\pi \leq \theta \leq \frac{\pi}{2} + k\pi$

R:



5. **(10 points)**
Show that if $|z| = 3$ then

$$\left|\frac{2\bar{z}^2 - z + 4i}{z+1}\right| \le \frac{33}{2}$$

By the triangle inequality

$$|2\bar{z}^2 - z + 4i| \le |2\bar{z}^2| + |z| + |4i|$$
$$= 2|\bar{z}|^2 + |z| + |4i|$$
$$= 2|z|^2 + |z| + |4i|$$
$$= 2 \cdot 3^2 + 3 + 4 = 25$$

By the other side of the triangle inequality

$$|z+1| \geq \big||z| - |1|\big|$$
$$= |3-1| = 2$$

$$\Rightarrow \quad \frac{1}{|z+1|} \le \frac{1}{2}.$$

~~finally~~, multiplying

$$\left|\frac{2\bar{z}^2 - z + 4i}{z+1}\right| \le \frac{25}{2}.$$

6. **(15 points)**
Evaluate each of the following limits, or state why it does not exist.

(a)

$$\lim_{z\to\infty} \frac{3z^2 + (1+2i)z - i}{iz^2 + 4}$$

this limit is $\frac{3}{i} = -3i$: $f(z) = \frac{3z^2 + (1+2i)z}{iz^2+4}$

$$\lim_{z\to 0} f\left(\frac{1}{z}\right) = \lim_{z\to 0} \frac{\frac{3}{z^2} + \frac{(1+2i)}{z} - i}{\frac{i}{z^2} + 4} \quad \leftarrow \text{multiply by } \frac{z^2}{z^2}$$

$$= \lim_{z\to 0} \frac{3 + (1+2i)z - iz^2}{i + 4z^2}$$

$$= \frac{3}{i} = -3i$$

(b)

$$\lim_{z\to\infty} (2z^2 + z - 3)$$

this limit is ∞ : $f(z) = 2z^2 + z - 3$

$$\lim_{z\to 0} \frac{1}{f\left(\frac{1}{z}\right)} = \lim_{z\to 0} \frac{1}{\frac{2}{z^2} + \frac{1}{z} - 3} \quad \text{mult. by } \frac{z^2}{z^2}$$

$$= \lim_{z\to 0} \frac{z^2}{2 + z - 3z^2}$$

$$= 0.$$

(c)

$$\lim_{z\to\infty} \frac{z}{\bar{z}}$$

This limit does not exist.

$$\lim_{z\to\infty} \frac{z}{\bar{z}} = \lim_{z\to 0} \frac{\bar{z}}{z} \Leftarrow$$ we saw in class that this limit does not exist.

<u>letting</u> $z = x$

$$\lim_{x\to 0} \frac{\bar{x}}{x} = 1$$

$$z = iy$$

$$\lim_{y\to 0} \frac{\overline{iy}}{iy} = -1$$

7. **(13 points)** Let

$$f(z) = \overbrace{\left(\frac{x^3}{3} + 2y\right)}^{u} + i\overbrace{\left(\frac{y^2}{2} - 2x\right)}^{v}$$

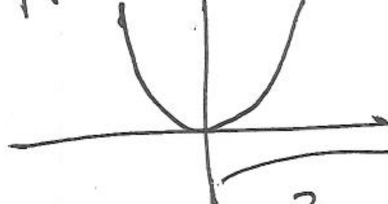
- Determine the set of points where the function $f(z)$ is differentiable, and calculate its derivative $f'(z)$ there.

<u>CR eq-ns</u>: $u_x = v_y \Rightarrow x^2 = y$

$u_y = -v_x \Rightarrow 2 = 2$ (automatically holds)

Since u_x, u_y, v_x, v_y are continuous everywhere, f will be differentiable when CR holds.

$\Rightarrow y = x^2$



<u>There</u>: $f'(z) = u_x + i v_x = \boxed{x^2 - 2i}$.

- Determine the set of points at which $f(z)$ is analytic. Explain your reasoning.

f is nowhere analytic. Any disk centered at points of the parabola $y = x^2$ will contain points where f is not differentiable.