MA 412 SAMPLE FINAL

(20 points) Question 1 Consider the function

$$f(z) = y^2 + i(3x^2)$$

(1) Find the values of $z = x + iy$ at which $f(z)$ is differentiable.
(2) State what it means for a general function $f(z)$ to be analytic at a point z_0.
(3) Where is the specific function $f(z)$, given above, analytic? Be sure you write down the Theorems you use to justify your answer.

(15 points) Question 2 Consider the mapping $f(z) = \exp(z)$. Find a region R such that f maps R onto the annulus $2 \leq |z| \leq 3$ in a one-to-one manner.

(20 points) Question 3 Using Σ notation, write down the Taylor series of the following functions at the points specified. In each case indicate the largest disk in which the expansion is valid.

(1)

$$f(z) = \frac{1}{(i+z)^2}$$

at $z_0 = 0$.

(2)

$$Log(1+z)$$

at $z_0 = 0$.

(10 points) Question 4 Using Σ notation, write down the Laurent series expansion of

$$f(z) = z^3 \sin(1/z)$$

at $z_0 = 0$. Is this singular point a pole or an essential singularity ?

(20 points) Question 5 Evaluate the following residues at the points specified

(1)

$$Res_{z=0} \frac{\exp(z)}{z^4}$$

(2)

$$Res_{z=0}\, z^3 \cos(1/z)$$

(15 points) Question 6 Evaluate the integral

$$\int_C \frac{dz}{z}$$

where C is the parabola $y = 2(x-1)^2$ joining $(1,0)$ to $(2,2)$.

(20 points) Question 7 Evaluate the integral

$$\int_C \frac{dz}{(z^2-9)(z^2+16)}$$

where:

(1) C is the circle $|z| = 1$ positively oriented.
(2) C is the circle $|z - 3i| = 2$ positively oriented.

(20 points) Question 8 Evaluate the following improper integral using the residue theorem. Please show all of your work.

$$\int_{-\infty}^{\infty} \frac{dx}{(x^4+1)}$$

(20) points Question 9 Find a formula for

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^n}$$

where $a > 0$ is a real number, and $n \geq 1$ is a positive integer.