Algebraic Curves, Twisted Vertex Operators, and Prym Varieties

by

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Abstract

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This dissertation consists of two parts. In the first, we extend the geometric vertex algebra formalism developed by Edward Frenkel to treat twisted vertex operators on algebraic curves. We develop the notion of conformal blocks for twisted vertex algebra modules, and show that in the Heisenberg case, these have a simple realization in terms of twisted coinvariants with respect to a twisted Heisenberg algebra. Furthermore, in the Heisenberg case, these conformal blocks naturally give rise to sheaves of twisted differential operators on Prym varieties.

In the second part, we apply the representation theory of vertex algebras and their twisted modules to construct Wakimoto modules for twisted affine algebras. Using the Wakimoto construction, we prove the Kac-Kazhdan conjecture on the characters of irreducible modules with generic critical highest weights in the twisted case. We provide explicit formulas for the twisted fields in the case of $A_2^{(2)}$.

Professor Edward Frenkel
Dissertation Committee Chair

To my Mother,

for showing me the value of learning,

and my Father,

for teaching me to love the beautiful things in life.

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Part I

First Part

Chapter 1

Introduction

Conformal field theory (CFT) in two dimensions provides a rich setting in which several areas of mathematics such as representation theory and algebraic geometry interact in a natural way. In recent years, much effort has been spent on setting up a precise mathematical framework for CFT. The algebraic aspect of the theory has been formalized in the language of vertex algebras (see [3, 16, 19, 14]). In order to understand the rich geometry behind CFT, this algebraic approach must be combined with a geometric formalism.

In [14], an algebro-geometric approach to vertex algebras is introduced (see [18, 2] for other approaches). Starting with a conformal vertex algebra V and an algebraic curve X, one can construct a vector bundle $\mathcal{V}_X$ on X such that vertex operators become (endomorphism-valued) sections of $\mathcal{V}_X^*$. This gives a coordinate-free description of vertex operators and allows one to make contact with the fascinating geometry pertaining to X and related moduli spaces.

In recent years, much attention in the physics literature has been focused on the so-called *orbifold models* (see [5, 4]). From the vertex algebra viewpoint, these models enlarge the theory by the inclusion of twisted modules (or twisted sectors as they are known in the physics literature). The precise mathematical definition of twisted modules has been given in [11, 6] and they have been extensively studied in recent years (see, e.g., [23, 7, 1]).

In this paper we extend the geometric formalism developed in [14] to twisted modules over vertex algebras. Let C be a smooth projective curve, and $H \subset Aut(C)$ a finite group of automorphisms of C. Suppose furthermore that V is a conformal vertex algebra, and that H acts on V by conformal automorphisms. We show that with this data, the vector bundle $\mathcal{V}_C$ acquires an H–equivariant structure, lifting the action of H on C. Let $X = C/H$ be the coarse quotient curve, and $\nu : C \to X$ the quotient map, ramified at the fixed points of H. Denote by $\overset{\circ}{C} \subset C$ (resp. $\overset{\circ}{X} \subset X$) the complement of the ramification points (resp. branch points) of ν, and by $\overset{\circ}{\nu} : \overset{\circ}{C} \to \overset{\circ}{X}$ the restriction of ν. Thus, $\overset{\circ}{C}$ is an H–principal bundle over $\overset{\circ}{X}$. The equivariant structure allows us to descend $\mathcal{V}_{\overset{\circ}{C}}$ to a vector bundle $\mathcal{V}^H_{\overset{\circ}{X}}$ on $\overset{\circ}{X}$. Explicitly,

$$\mathcal{V}^H_{\overset{\circ}{X}} = Aut_{\overset{\circ}{C}} \underset{H \times \mathrm{Aut}\,\mathcal{O}}{\times} V$$

Let $x \in X$. Then x corresponds to an H–orbit $\mathcal{O}_x$ on C. For each point $p \in \nu^{-1}(x)$, the stabilizer H_p is a (possibly trivial) cyclic group, which has a canonical generator h_p, the monodromy around p. We show that to a collection $\{M_p^{h_p}\}_{p \in \nu^{-1}(x)}$ of h_p–twisted modules satisfying certain properties, there corresponds a section $\mathcal{Y}^{\mathcal{M}_x}$ of $\mathcal{V}^{H,*}_{\overset{\circ}{X}}$ on $\mathcal{D}_x^\times$. We call the collection $\{M_p^{h_p}\}$ a V–module along $\nu^{-1}(x)$. Note that when $\nu^{-1}(x)$ consists of $|H|$ points, $h_p = e$, and $M_p^{h_p}$ are ordinary V–modules.

Using this structure we define the spaces of conformal blocks in the twisted setting. The space of conformal blocks is attached to a pair (C, H) as above, a collection of V–modules along $\nu^{-1}(x)$ attached to all points of $X \backslash \overset{\circ}{X}$, and a (possibly empty) collection V–modules along $\nu^{-1}(x), x \in \overset{\circ}{X}$. We give two equivalent definitions of the space of conformal blocks: using the action of a certain Lie algebra obtained from Fourier coefficients of vertex operators, and using analytic continuation (as in [14]). In the case of the Heisenberg vertex algebra this definition may be simplified using twisted versions of the Heisenberg Lie algebra.

Chapter 2

Vertex Algebras and Twisted Modules

In this chapter, we review some basics concerning affine Lie algebras, vertex algebras, and their modules. We follow the treatment presented in [14].

2.1 Affine Lie algebras

Let $\mathfrak{k}$ be a complex Lie algebra, and $(\cdot,\cdot)$ an invariant bilinear form on $\mathfrak{k}$.

Definition 2.1.1. The *affine Lie algebra* associated to $\mathfrak{k}, (\cdot,\cdot)$, is the Lie algebra

$$\hat{\mathfrak{k}} = \mathfrak{k} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} \cdot K$$

with commutation relations

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t) + Res_{t=0} f(t)dg(t) \cdot K$$

Example 2.1.1. Let $\mathfrak{k} = \mathbb{C}$, with $(A, B) = AB$, then $\hat{k} = \mathcal{H}$, the Heisenberg Lie algebra, with generators $b_n = t^n$, satisfying $[b_n, b_m] = n\delta_{n,-m}K$.

Example 2.1.2. Let $\mathfrak{k} = \mathfrak{g}$, a finite dimensional, simple Lie algebra, and $(\cdot,\cdot) = \frac{1}{2h^\vee}(\cdot,\cdot)_K$, where the $(\cdot,\cdot)_K$ is the Killing form. Then $\hat{k} = \hat{\mathfrak{g}}$, the affine Kac-Moody algebra associated to $\mathfrak{g}$, with generators $X_n = X \otimes t^n, X \in \mathfrak{g}$, subject to $[X_n, Y_m] = [X,Y]_{n+m} + (X,Y)\,\delta_{n,-m}K$

2.2 Twisted Affine Algebras

The construction in the last section can be "twisted" by the presence of a finite-order automorphism σ of $\mathfrak{k}$, preserving the bilinear form $(\cdot,\cdot)$.

Definition 2.2.1. Let σ be an automorphism of $\mathfrak{k}$ of order N preserving $(\cdot,\cdot)$. The *twisted affine Lie algebra* associated to $(\mathfrak{k},\sigma)$ is the Lie algebra

$$\hat{\mathfrak{k}}^{\sigma} = \oplus \mathfrak{k}_j \otimes t^{\frac{j}{N}}\mathbb{C}[t,t^{-1}] \oplus \mathbb{C}\cdot K$$

where $\mathfrak{k}_j$ denotes the $\exp\frac{2\pi i j}{N}$ eigenspace of σ, with commutation relations

$$[A \otimes f(t^{\frac{1}{N}}), B \otimes g(t^{\frac{1}{N}})] = [A,B] \otimes f(t^{\frac{1}{N}})g(t^{\frac{1}{N}}) + Res_{t=0} f(t^{\frac{1}{N}})dg(t^{\frac{1}{N}}) \cdot K$$

Example 2.2.1. Let $\mathfrak{k} = \mathbb{C}$, and $\sigma(z) = -z$. Then $\hat{\mathfrak{k}}^{\sigma} = \mathcal{H}^{\sigma}$, the twisted Heisenberg algebra, with generators $b_n = t^n, n \in \frac{1}{2} + \mathbb{Z}$, subject to $[b_n, b_m] = n\delta_{n,-m}K$

Example 2.2.2. Let $\mathfrak{k} = \mathfrak{g}$ as above, and σ a nontrivial outer automorphism of $\mathfrak{g}$ (equivalently, a non-trivial Dynkin diagram automorphism). Then $\hat{\mathfrak{k}}^{\sigma} = \hat{\mathfrak{g}}^{\sigma}$, the twisted affine Kac-Moody Lie algebra.

2.3 Vertex Algebras

Definition 2.3.1. A *vertex algebra* is a collection of data:

- (*space of states*) a $\mathbb{Z}_+$ - graded vector space

$$V = \bigoplus_{m=0}^{\infty} V_m$$

  with $dim(V_m) < \infty$;

- (*vacuum vector*) a vector $|0\rangle \in V_0$;
- (*translation operator*) a linear operator $T : V \to V$ of degree one;
- (*vertex operators*) a linear operation

  $$Y(\cdot, z) : V \to \operatorname{End} V[[z^{\pm 1}]]$$

  taking each $A \in V_m$ to a field

  $$Y(A, z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1}$$

  of conformal dimension m (i.e. $degA_n = -n + m - 1$).

These data are subject to the following axioms:

- (*vacuum axiom*) $Y(|0\rangle, z) = \mathrm{Id_V}$. Furthermore, for any $A \in V$ we have

  $$Y(A, z)|0\rangle \in V[[z]]$$

  so that $Y(A, z)|0\rangle$ has a well-defined value at $z = 0$, and

  $$Y(A, z)|0\rangle|_z = A.$$

  In other words, $A_n|0\rangle = 0, n \geq 0$, and $A_{-1}|0\rangle = A$
- (*translation axiom*) For any $A \in V$,

  $$[T, Y(A, z)] = \partial_z Y(A, z)$$

  and $T|0\rangle = 0$.
- (*locality axiom*) For any $A, B \in V$, there exists $N \in \mathbb{Z}_+$ such that

  $$(z - w)^N [Y(A, z), Y(B, w)] = 0$$

  as formal power series in $\operatorname{End} V[[z^{\pm 1}, w^{\pm 1}]]$

Definition 2.3.2. We will say that a vertex algebra V is *conformal*, if there exists a vector $\omega \in V_2$ such that the Fourier coefficients of the field

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

satisfy the commutation relations of the Virasoro algebra.

Example 2.3.1. Let $\mathcal{H}$ be as in Example 2.1.1, $\mathcal{H}_+$ the Lie subalgebra of $\mathcal{H}$ generated by $b_n, n \geq 0$. Let $\mathbb{C}_k$ denote the one-dimensional representation of $\mathcal{H}_+ \oplus \mathbb{C} \cdot K$ where K acts by k and all other elements act by 0. Then $\pi_k = \mathrm{Ind}_{\mathcal{H}_+ \oplus \mathbb{C} \cdot K}^{\mathcal{H}} \mathbb{C}_k$ has the structure of a vertex algebra, and will be henceforth referred to as the Heisenberg vertex algebra of level k.

Example 2.3.2. Let $\mathfrak{g}, \hat{\mathfrak{g}}$ be as in Example 2.1.2, and let $\mathbb{C}_k$ denote the one-dimensional representation of $\hat{\mathfrak{g}}_+ = \mathfrak{g}[[t]] \oplus \mathbb{C} \cdot K$ where the first factor acts by 0, and K by multiplication by k. Then the "vacuum module" $V_k(g) = \mathrm{Ind}_{\hat{\mathfrak{g}}_+}^{\hat{\mathfrak{g}}} \mathbb{C}_k$ has the structure of a vertex algebra.

2.4 Modules

Definition 2.4.1. A vector space M is called a V–module if it is equipped with an operation

$$Y^M : V \to \mathrm{End}\, M[[z^{\pm 1}]],$$
$$A \mapsto Y^M(A, z) = \sum_{n\in\mathbb{Z}} A^M_{(n)} z^{-n-1}$$

such that for any $v \in M$ we have $A^M_{(n)} v = 0$ for large enough n. This operation must satisfy the following axioms:

- $Y^M(|0\rangle, z) = \mathrm{Id}_M$;

- For any $v \in M$ there exists an element

$$f_v \in M[[z, w]][z^{-1}, w^{-1}, (z-w)^{-1}]$$

such that the formal power series

$$Y^M(A,z)Y^M(B,w)v \qquad \text{and} \qquad Y^M(Y(A,z-w)B,w)v$$

are expansions of f_v in $M((z))((w))$ and $M((w))((z-w))$, respectively.

Remark 1. The Fourier coefficients of the field $Y^M(\omega, z)$ generate an action of the Virasoro algebra on M.

Definition 2.4.2. We will call M an *admissible* module for V if the induced L_0^M–gradation is integral.

Example 2.4.1. Let $V_k(\mathfrak{g})$ be as in 2.3.2. Then $V_k(\mathfrak{g})$ – modules are precisely the category $\mathcal{O}$ $\hat{\mathfrak{g}}$ modules.

2.5 Twisted Modules

Now let σ_V be a conformal automorphism of V, i.e., an automorphism of the underlying vector space preserving all of the above structures (so in particular $\sigma_V(\omega) = \omega$). We will assume that σ_V has finite order $N > 1$.

Definition 2.5.1. A vector space M^σ is called a σ_V*–twisted* V–module (or simply twisted module) if it is equipped with an operation

$$\begin{aligned} Y^{M^\sigma} &: V \to \operatorname{End} M^\sigma[[z^{\pm\frac{1}{N}}]], \\ A &\mapsto Y^{M^\sigma}(A, z^{\frac{1}{N}}) = \sum_{n\in\frac{1}{N}\mathbb{Z}} A_{(n)}^{M^\sigma} z^{-n-1} \end{aligned}$$

such that for any $v \in M^\sigma$ we have $A_{(n)}^{M^\sigma} v = 0$ for large enough n. Please note that we use the notation $Y^{M^\sigma}(A, z^{\frac{1}{N}})$ rather than $Y^{M^\sigma}(A,z)$ in the twisted setting. This operation must satisfy the following axioms:

- $Y^{M^\sigma}(|0\rangle, z^{\frac{1}{N}}) = \mathrm{Id}_{M^\sigma}$

- For any $v \in M^\sigma$, there exists an element

$$f_v \in M^\sigma[[z^{\frac{1}{N}}, w^{\frac{1}{N}}]][z^{-\frac{1}{N}}, w^{-\frac{1}{N}}, (z-w)^{-1}]$$

  such that the formal power series

$$Y^{M^\sigma}(A, z^{\frac{1}{N}}) Y^{M^\sigma}(B, w^{\frac{1}{N}}) v \qquad \text{and} \qquad Y^{M^\sigma}(Y(A, z-w)B, w^{\frac{1}{N}}) v$$

  are expansions of f_v in $M^\sigma((z^{\frac{1}{N}}))((w^{\frac{1}{N}}))$ and $M^\sigma((w^{\frac{1}{N}}))((z-w))$, respectively.

- If $A \in V$ is such that $\sigma_V(A) = e^{\frac{2\pi i m}{N}} A$, then $A^{M^\sigma}_{(n)} = 0$ unless $n \in \frac{m}{N} + \mathbb{Z}$.

The series $Y^{M^\sigma}(A, z)$ are called twisted vertex operators.

Remark 2. The Fourier coefficients of the field $Y^{M^\sigma}(\omega, z)$ generate an action of the Virasoro algebra on M^σ.

Definition 2.5.2. We will call M^σ *admissible*, if the induced $L_0^{M^\sigma}$–gradation takes values in $\frac{1}{N}\mathbb{Z}$.

Example 2.5.1. Let $\hat{\mathfrak{g}}^\sigma$ be as in 2.2.2, and let σ_V denote the vertex algebra automorphism of $V_k(\mathfrak{g})$ induced from σ. Then the σ_V - twisted $V_k(\mathfrak{g})$ modules are precisely the Category $\mathcal{O}$ $\hat{\mathfrak{g}}^\sigma$ - modules.

Chapter 3

A Coordinate-Free Approach to Twisted Vertex Operators

3.1 Torsors and Twists

Let M^σ be an admissible conformal σ_V–twisted V–module where $\mathrm{ord}(\sigma_V) = N$. In this section we define a group $\mathrm{Aut}_N \mathcal{O}$ which naturally acts on M^σ, as well as natural torsors for $\mathrm{Aut}_N \mathcal{O}$. This will allow us to twist M^σ by a certain torsor of formal coordinates.

3.1.1 The Group $\mathrm{Aut}_N \mathcal{O}$

Let $\mathrm{Aut}\,\mathbb{C}[[z^{\frac{1}{N}}]]$ denote the group of continuous algebra automorphisms of $\mathbb{C}[[z^{\frac{1}{N}}]]$. Since $\mathbb{C}[[z^{\frac{1}{N}}]]$ is topologically generated by $z^{\frac{1}{N}}$, an automorphism ρ of $\mathbb{C}[[z^{\frac{1}{N}}]]$ is completely determined by the image of $z^{\frac{1}{N}}$, which is a series of the form

$$\rho(z^{\frac{1}{N}}) = \sum_{n \in \frac{1}{N}\mathbb{Z}, n>0} c_n z^n, \tag{3.1}$$

where $c_{\frac{1}{N}} \neq 0$. Hence we identify $\mathrm{Aut}\,\mathbb{C}[[z^{\frac{1}{N}}]]$ with the space of power series in $z^{\frac{1}{N}}$ having non-zero linear term. For more on the structure of the group $\mathrm{Aut}\,\mathbb{C}[[z^{\frac{1}{N}}]]$, see Section 5.1 of [14]. Recall that we denote $\mathbb{C}[[z]]$ by $\mathcal{O}$.

Definition 3.1.1. $\mathrm{Aut}_N \mathcal{O}$ is the subgroup of $\mathrm{Aut}\,\mathbb{C}[[z^{\frac{1}{N}}]]$ preserving the subalgebra $\mathbb{C}[[z]] \subset \mathbb{C}[[z^{\frac{1}{N}}]]$.

Thus, $\mathrm{Aut}_N \mathcal{O}$ consists of power series of the form

$$\rho(z^{\frac{1}{N}}) = \sum_{n \in \frac{1}{N}+\mathbb{Z}, n>0} c_n z^n, \qquad c_{\frac{1}{N}} \neq 0. \tag{3.2}$$

There is a homomorphism $\mu : \mathrm{Aut}_N \mathcal{O} \to \mathrm{Aut}\,\mathcal{O}$ which takes $\rho \in \mathbb{C}[[z^{\frac{1}{N}}]]$ to the automorphism of $\mathbb{C}[[z]]$ that it induces. At the level of power series, this is just the map $\mu : \rho(z) \mapsto \rho(z)^N$. The kernel consists of the automorphisms of the form $z^{\frac{1}{N}} \mapsto \epsilon z^{\frac{1}{N}}$, where ϵ is an Nth root of unity, so we have the following exact sequence:

$$1 \to \mathbb{Z}/N\mathbb{Z} \to \mathrm{Aut}_N \mathcal{O} \to \mathrm{Aut}\,\mathcal{O} \to 1\,.$$

The Lie algebra of $\mathrm{Aut}\,\mathbb{C}[[z^{\frac{1}{N}}]]$ is

$$\mathrm{Der}^{(0)}\,\mathbb{C}[[z^{\frac{1}{N}}]] = z^{\frac{1}{N}}\mathbb{C}[[z^{\frac{1}{N}}]]\partial_{z^{\frac{1}{N}}},$$

and the Lie algebra of $\mathrm{Aut}_N \mathcal{O}$ is its Lie subalgebra $\mathrm{Der}_N^{(0)}\,\mathcal{O} = z^{\frac{1}{N}}\mathbb{C}[[z]]\partial_{z^{\frac{1}{N}}}$. The homomorphism μ induces an isomorphism of the corresponding Lie algebras sending

$$z^{k+\frac{1}{N}}\partial_{z^{\frac{1}{N}}} \mapsto N z^k \partial_z, \qquad k \in \mathbb{Z}, k \geq 0.$$

3.1.2 The $\mathrm{Aut}_N \mathcal{O}$–Torsor of Special Coordinates

We use the notation of Appendix A. Let $(\mathcal{D}, \sigma_{\mathcal{D}})$ be a pair consisting of a disc $\mathcal{D} = \mathrm{Spec}\,R$ and an automorphism $\sigma_{\mathcal{D}}$ of $\mathcal{D}$ (equivalently, of R) of order N. We denote by $\overline{\mathcal{D}}$ the quotient of $\mathcal{D}$ by $\langle \sigma_{\mathcal{D}} \rangle$, i.e., the disc $\mathrm{Spec}\,R^{\sigma_{\mathcal{D}}}$, where $R^{\sigma_{\mathcal{D}}}$ is the subalgebra of $\sigma_{\mathcal{D}}$–invariant elements.

A formal coordinate t is called a *special coordinate* with respect to σ if $\sigma(t) = \epsilon t$, where ϵ is an Nth root of unity, or equivalently, if t^N is a formal coordinate on $\overline{\mathcal{D}}$. We denote by $\mathcal{A}ut(\mathcal{D})$ the set of all formal coordinates on $\mathcal{D}$ and by $\mathcal{A}ut_N(\mathcal{D})$ the subset of $\mathcal{A}ut(\mathcal{D})$ consisting of special formal coordinates. The set $\mathcal{A}ut_N(\mathcal{D})$ carries a simply transitive right action of the group $\mathrm{Aut}_N \mathcal{O}$ given by $t \mapsto \rho(t)$, where ρ is the power series given in (3.2), i.e., $\mathcal{A}ut_N(\mathcal{D})$ is an Aut_N–torsor.

3.1.3 Twisting Modules by $\mathcal{A}ut_N(\mathcal{D})$

Let M^σ be a σ_V–twisted admissible module over a conformal vertex algebra V. Define a representation r^{M^σ} of the Lie algebra $\mathrm{Der}_N^{(0)}\,\mathcal{O}$ on M^σ by the formula

$$z^{k+\frac{1}{N}}\partial_{z^{\frac{1}{N}}} \to -N \cdot L_k^{M^\sigma}.$$

It follows from the definition of a twisted module that the operators $L_k^{M^\sigma}, k > 0$, act locally nilpotently on M^σ and that the eigenvalues of $L_0^{M^\sigma}$ lie in $\frac{1}{N}\mathbb{Z}$, so that the operator $N \cdot L_0^{M^\sigma}$ has integer eigenvalues. This implies that the Lie algebra representation r^{M^σ} may be exponentiated to a representation R^{M^σ} of the group $\mathrm{Aut}_N\,\mathcal{O}$.

We now twist the module M^σ by the action of $\mathrm{Aut}_N\,\mathcal{O}$ and define the vector space

$$\mathcal{M}^\sigma(\mathcal{D}) \stackrel{\mathrm{def}}{=} \mathcal{A}ut_N(\mathcal{D}) \underset{\mathrm{Aut}_N\,\mathcal{O}}{\times} M^\sigma. \tag{3.3}$$

Thus, vectors in $\mathcal{M}^\sigma(\mathcal{D})$ are pairs (t, v), up to the equivalence relation

$$(\rho(t), v) \sim (t, R^{M^\sigma}(v)), \qquad t \in \mathcal{A}ut_N(\mathcal{D}), v \in M^\sigma.$$

When $\mathcal{D}$=$\mathcal{D}_x$, the formal neighborhood of a point x on an algebraic curve X, we will use the notation $\mathcal{M}^\sigma{}_x$.

3.2 Twisted Vertex Operators as Sections

Our goal is to give a coordinate-independent description of the operation Y^{M^σ}. In order to do this we need to find how the operation Y^{M^σ} transforms under changes of special coordinates. This is the subject of this section. Henceforth, we shorten "Twisted Vertex Operator" to TVOP.

3.2.1 The Transformation Formula for TVOPs

Let $\mathcal{O} = \mathbb{C}[[t]]$. Denote by R^V the representation of the group $\mathrm{Aut}\,\mathcal{O}$ on V obtained by exponentiating the representation r^V of the Lie algebra $\mathrm{Der}^{(0)}\,\mathcal{O}$ sending $t^{n+1}\partial_t$ to $-L_n, n \geq 0$ (see Section 5.2 of [14]). Recall that for any $\rho(t^{\frac{1}{N}}) \in \mathrm{Aut}_N\,\mathcal{O}$, we have

$\rho(t^{\frac{1}{N}})^N \in \operatorname{Aut} \mathcal{O}$. For any $\tau(t) \in \operatorname{Aut} \mathcal{O}$ we denote by τ_z the element of $\operatorname{Aut}(\mathbb{C}[[z]]\widehat{\otimes}\mathcal{O})$ obtained by expanding $\tau(z+t) - \tau(z)$ in powers of t (see Section 5.4.5 of [14]). Then we have the following analogue of Lemma 5.4.6 from [14] (that lemma is originally due to Y.-Z. Huang [18]).

Lemma 3.2.1. *For any $A \in V$, $\rho \in \operatorname{Aut}_N \mathcal{O}$*

$$R^{M^\sigma}(\rho) Y^{M^\sigma}(R^V((\rho^N)_z)^{-1} A, \rho(z^{\frac{1}{N}})) R^{M^\sigma}(\rho)^{-1} = Y^{M^\sigma}(A, z^{\frac{1}{N}}). \tag{3.4}$$

Proof. The exponential map $\operatorname{Der}_N^{(0)} \to \operatorname{Aut}_N \mathcal{O}$ is surjective, so it suffices to consider the infinitesimal version of (3.4). Write

$$\rho = \exp\left(\epsilon v(z^{\frac{1}{N}}) \partial_{z^{\frac{1}{N}}}\right) \cdot z^{\frac{1}{N}},$$

where

$$v(z^{\frac{1}{N}}) = - \sum_{k \in \mathbb{Z}, k \geq 0} v_k z^{k + \frac{1}{N}}.$$

We have

$$\rho^N = \exp\left(\epsilon u(z) \partial_z\right) \cdot z,$$

where

$$u(z) = -N \sum_{k \in \mathbb{Z}, k \geq 0} v_k z^{k+1}.$$

To check that formula (3.4) holds, it suffices to check that the ϵ–linear term in it vanishes. Denote $r^{M^\sigma}(v(z^{\frac{1}{N}})\partial_{z^{\frac{1}{N}}})$ by $r_v^{M^\sigma}$ and the ϵ–linear term in $R^V((\rho^N)_z)$ by $r_{u,z}^V$. The ϵ–linear term in (3.4) reads

$$\begin{aligned}
&(Id + \epsilon r_v^{M^\sigma}) Y^{M^\sigma}((Id - \epsilon r_{u,z}^V)A, z^{\frac{1}{N}} + \epsilon v(z^{\frac{1}{N}}))(Id - \epsilon r_v^{M^\sigma}) - Y^{M^\sigma}(A, z^{\frac{1}{N}}) \\
&= \epsilon [r_v^{M^\sigma}, Y^{M^\sigma}(a, z^{\frac{1}{N}})] - \epsilon Y^{M^\sigma}(r_{u,z}^V \cdot A, z^{\frac{1}{N}}) + \epsilon v(z^{\frac{1}{N}}) \partial_{z^{\frac{1}{N}}} Y^{M^\sigma}(A, z^{\frac{1}{N}}).
\end{aligned} \tag{3.5}$$

We find that

$$r_{u,z}^V \cdot A = - \sum_{m \geq 0} \frac{1}{(m+1)!} (\partial_z^{m+1} u(z)) L_m \cdot A$$

and

$$r_v^{M^\sigma} = N \sum_{m \in \mathbb{Z}, m \geq 0} v_m L_m^{M^\sigma},$$

so that vanishing of (3.5) is equivalent to the identity

$$[r_v^{M^\sigma}, Y^{M^\sigma}(A, z^{\frac{1}{N}})] = -\sum_{m\geq -1} \frac{1}{(m+1)!}(\partial_z^{m+1}u(z))Y^{M^\sigma}(L_m \cdot A, z^{\frac{1}{N}}).$$

Since

$$v(z^{\frac{1}{N}})\partial_{z^{\frac{1}{N}}} = u(z)\partial_z,$$

this identity follows from the OPE between a twisted vertex operator and the Virasoro field in the same way as in Section 5.2.3 of [14]. □

3.2.2 Example: primary fields

Recall that a vector $A \in V$ is called a primary vector of conformal dimension Δ if it satisfies

$$L_n A = 0, \quad n > 0; \qquad L_0 A = \Delta A.$$

As shown in Lemma 5.3.4 of [14], the corresponding vertex operator $Y(A, z)$ transforms under coordinate changes as an endomorphism-valued Δ–differential on the punctured disc. Now formula (3.4) implies an analogous transformation formula for the corresponding twisted vertex operator $Y^{M^\sigma}(A, z^{\frac{1}{N}})$.

Corollary 3.2.1. *Let $A \in V$ be a primary vector of conformal dimension Δ, and $\rho \in \mathrm{Aut}_N \mathcal{O}$. Then*

$$R^{M^\sigma}(\rho)Y^{M^\sigma}(A, \rho(z^{\frac{1}{N}}))R^{M^\sigma}(\rho)^{-1}\left(\partial_z(\rho^N(z^{\frac{1}{N}}))\right)^\Delta = Y^{M^\sigma}(A, z^{\frac{1}{N}}). \tag{3.6}$$

3.3 Coordinate-Independent Interpretation of TVOPs

3.3.1 Recollections from [14]

Let X be a smooth curve and $\mathcal{A}ut_X$ the principal $\mathrm{Aut}\,\mathcal{O}$–bundle of formal coordinates on X. The fiber of $\mathcal{A}ut_X$ at $x \in X$ is the $\mathrm{Aut}\,\mathcal{O}$–torsor $\mathcal{A}ut_x$ of formal coordinates at x (see Section 5.4 of [14] for details). Given a conformal vertex algebra V, set

$$\mathcal{V} = \mathcal{V}_X = \mathcal{A}ut_X \underset{\mathrm{Aut}\,\mathcal{O}}{\times} V. \tag{3.7}$$

This is a vector bundle whose fiber at $x \in X$ is the $\mathcal{A}ut_x$–twist of V,

$$\mathcal{V}_x = \mathcal{A}ut_x \underset{\mathrm{Aut}\,\mathcal{O}}{\times} V.$$

i.e., the set of pairs (z, A) where z is a formal coordinate at x and $A \in V$, modulo the equivalence condition $(\rho(z), A) \sim (z, R^V(\rho) \cdot A)$ for $\rho \in \mathrm{Aut}\,\mathcal{O}$.

As explained in Chapter 5 of [14], for any $x \in X$ the vertex operation Y gives rise to a canonical section $\mathcal{Y}_x$ of the dual bundle $\mathcal{V}^*$ on the punctured disc $\mathcal{D}_x^\times$ with values in $\mathrm{End}\,\mathcal{V}_x$. Equivalently, we have a canonical linear map

$$\mathcal{Y}_x^\vee : \Gamma(\mathcal{D}_x^\times, \mathcal{V} \otimes \Omega_X) \to \mathrm{End}\,\mathcal{V}_x, \qquad s \mapsto \mathrm{Res}_x \langle \mathcal{Y}_x, s \rangle.$$

If we choose a formal coordinate z at x and use it to trivialize $\mathcal{V}|_{\mathcal{D}_x}$, then $\mathcal{Y}_x^\vee$ is given by the formula

$$\mathcal{Y}_x^\vee(A \otimes z^n dz) = A_{(n)}.$$

Furthermore, the map $\mathcal{Y}_x^\vee$ factors through the quotient

$$U(\mathcal{V}_x) = \Gamma(\mathcal{D}_x^\times, \mathcal{V} \otimes \Omega_X)/\,\mathrm{Im}\,\nabla.$$

The latter is a Lie algebra and the resulting map $U(\mathcal{V}_x) \to \mathrm{End}\,\mathcal{V}_x$ is a Lie algebra homomorphism (see Section 8.2 of [14]).

More generally, let M be an admissible V–module. We attach to it a vector bundle $\mathcal{M}$ on X in the same way as above. The module operation Y^M then gives rise to a canonical section $\mathcal{Y}_x^M$ of $\mathcal{V}^*|_{\mathcal{D}_x^\times}$ with values in $\mathrm{End}\,\mathcal{M}_x$. Equivalently, we have a canonical linear map

$$\mathcal{Y}_x^{M,\vee} : \Gamma(\mathcal{D}_x^\times, \mathcal{V} \otimes \Omega_X) \to \mathrm{End}\,\mathcal{M}_x,$$

which factors through $U(\mathcal{V}_x)$ (see Section 6.3.6 of [14]).

In this section we obtain analogous results for twisted modules over vertex algebras.

3.3.2 The Vector Bundle $\mathcal{V}_X^H$

Let C be a smooth projective curve, and $H \subset Aut(C)$ a finite group of automorphisms of C. Suppose furthermore that V is a conformal vertex algebra, and that H acts on V by conformal automorphisms. The vector bundle $\mathcal{V}_C$ carries an H–equivariant structure lifting the action of H on C. It is given by

$$h \cdot (p, (A, z)) \stackrel{def}{=} (h(p), (h(A), z \circ h^{-1})) \tag{3.8}$$

where $z \circ h^{-1}$ is the coordinate induced at $h(p)$ from z. Let $X = C/H$ be the coarse quotient curve, and $\nu : C \to X$ the quotient map, ramified at the points where H has non-trivial stabilizers. Denote by $\overset{\circ}{C} \subset C$ (resp. $\overset{\circ}{X} \subset X$) the complement of the ramification points (resp. branch points) of ν, and by $\overset{\circ}{\nu} : \overset{\circ}{C} \to \overset{\circ}{X}$ the restriction of ν. Thus, $\overset{\circ}{C}$ is an H–principal bundle over $\overset{\circ}{X}$. The action of H on $\overset{\circ}{C}$ is free, and $\mathcal{V}_{\overset{\circ}{C}}$ descends to a vector bundle $\mathcal{V}^H_{\overset{\circ}{X}}$ on $\overset{\circ}{X}$. More explicitly,

$$\mathcal{V}^H_{\overset{\circ}{X}} = Aut_{\overset{\circ}{C}} \underset{\mathrm{Aut}\,\mathcal{O}\times H}{\times} V$$

Here, H acts on $Aut_{\overset{\circ}{C}}$ by $h(p, z) = (h(p), z \circ h^{-1})$, and this action commutes with the action of $\mathrm{Aut}\,\mathcal{O}$. The actions of H and $\mathrm{Aut}\,\mathcal{O}$ on V commute because H is a conformal automorphism of V, and thus commutes with the Virasoro action.

$\mathcal{V}^H_{\overset{\circ}{X}}$ possesses a flat connection ∇^H. If z is a local coordinate $x \in \overset{\circ}{X}$, ∇^H is given by the expression $d + L^V_{-1} \otimes dz$.

3.3.3 Modules along H–orbits

Let $x \in X$. Then every point $p \in \nu^{-1}(x)$ has a (possibly trivial) cyclic stabilizer of order N, which we denote H_p. Each H_p has a canonical generator h_p, which corresponds to the monodromy of a small loop around x. Suppose that we are given the following data:

1. A collection of conformal V–modules $\{M^{h_p}_p\}_{p\in\nu^{-1}(x)}$, one for each point in the fiber, such that $M^{h_p}_p$ is h_p–twisted.

2. A collection of maps $T_{g,p,g(p)} : M^{h_p}_p \mapsto M^{h_{g(p)}}_{g(p)}$, $g \in H$, $p \in \nu^{-1}(x)$ satisfying

$$T_{gk,p,gk(p)} = T_{g,k(p),gk(p)} \circ T_{k,p,k(p)}$$

$$T^{-1}_{g,p,g(p)} = T_{g^{-1},g(p),p}$$

and

$$T_{g,p,g(p)}^{-1} Y^{M_{g(p)}^{h_{g(p)}}}(g \cdot A, z) T_{g,p,g(p)} = Y^{M_p^{h_p}}(A, z)$$

3. The T's commute with the action of $\mathrm{Aut}_N \mathcal{O}$

4. If $g \in H_p$, the action of $T_{g,p,p}$ on $M_p^{h_p}$ agrees with the action of $\mathrm{Aut}_N \mathcal{O}$ via $g(z^{\frac{1}{N}}) = z^{\frac{1}{N}} \circ g^{-1}$

Given a collection $\{M_p^{h_p}\}_{p \in \nu^{-1}(x)}$, we can form its $\mathrm{Aut}_N \mathcal{O}$–twist, which is the collection $\mathcal{M}_p^{h_p}(\mathcal{D}_p)$, together with the T–operators. Finally, we observe that the T–operators induce an equivalence relation on $\bigcup_{p \in \nu^{-1}(x)} \mathcal{M}_p^{h_p}(\mathcal{D}_p)$. We denote the set of equivalence classes by $\mathcal{M}_x$.

Definition 3.3.1. We call $\mathcal{M}_x$ a *V–module along $\nu^{-1}(x)$*

Example 3.3.1. Suppose that we are given an ordinary (untwisted) V–module M together with an action of H such that $g^{-1} Y^M(g \cdot A, z) g = Y^M(A, z), \forall g \in H$ (in particular, this holds for $M = V$). We show that if $x \in \overset{\circ}{X}$ (i.e. the fiber over x contains $|H|$ points), then M gives rise to a V–module along $\nu^{-1}(x)$ with the following data: for $p \in \nu^{-1}(x)$, define $M_p^e = M$, and let $T_{g,p,g(p)} = g$. The requirements of definition 3.3.1 are easily checked.

Example 3.3.2. Suppose that H is cyclic, and that $\nu^{-1}(x)$ consists of a single point p. Let $h_p \in H_p = H$ be as above. We show that an admissible h_p–twisted module M^{h_p} gives rise to a module along $\nu^{-1}(x)$. Let $M_p^{h_p} = M^{h_p}$. We observe that H embeds into $\mathrm{Aut}_N \mathcal{O}$ by the action $h(z^{\frac{1}{N}}) = z^{\frac{1}{N}} \circ h^{-1}$, and define $T_{h_p,p,p}$ via the representation $R^{M^{h_p}}$ of $\mathrm{Aut}_N \mathcal{O}$ on M^{h_p}.

3.3.4 TVOPs as sections of $\mathcal{V}_X^{H,*}$

We begin with the observation that a section of $\mathcal{V}_{\overset{\circ}{X}}^{H}$ over $U \subset \overset{\circ}{X}$ is the same as an H–equivariant section of $\mathcal{V}_{\overset{\circ}{C}}$ over $\nu^{-1}(U)$, and likewise for $\mathcal{V}_{\overset{\circ}{X}}^{H,*}$. Transposing this observation to the formal setting, we see that to define a section of $\mathcal{V}_{\overset{\circ}{X}}^{H,*}$ on $\mathcal{D}_x^{\times}$ is to define an H–equivariant section of $\mathcal{V}_{\overset{\circ}{C}}^{*}$ on $\coprod_{p \in \nu^{-1}(x)} \mathcal{D}_p^{\times}$.

Remark 3. Even when $x \in X \backslash \overset{\circ}{X}$, the completion $\mathcal{V}^H_{\overset{\circ}{X}}|_{\mathcal{D}_x^\times}$ exists, so we may speak of the restriction of $\mathcal{V}^H_{\overset{\circ}{X}}$ to $\mathcal{D}_x^\times$ in the sense of Appendix A.

Let $p \in \nu^{-1}(x)$, and let $z_p^{\frac{1}{N}}$ be an h_p–special coordinate at p which gives an actual (analytic) coordinate in a neighborhood W of p. Then if $q \in W, q \neq p$, $z_p - z_p(q)$ is a coordinate at q. This implies that the map

$$\iota_{z_p} : V \times \mathcal{D}_p^\times \mapsto \mathcal{V}_{\overset{\circ}{C}}|_{\mathcal{D}_p^\times}$$

given by

$$\iota_{z_p}(A, q) = (A, z_p - z_p(q))$$

is a trivialization of $\mathcal{V}_{\overset{\circ}{C}}$ on $\mathcal{D}_p^\times$.

Theorem 3.3.1. *Let $x \in X$, and $\mathcal{M}_x$ a V–module along $\nu^{-1}(x)$. For each $p \in \nu^{-1}(x)$, choose an h_p–special coordinate $z_p^{\frac{1}{N}}$ at p. Define an* $\mathrm{End}(\mathcal{M}_x)$*–valued section $\mathcal{Y}^{\mathcal{M}_x}$ of $\mathcal{V}^*_{\overset{\circ}{C}}$ on $\coprod_{p\in\nu^{-1}(x)} \mathcal{D}_p^\times$ by the formula*

$$\langle (z_p^{\frac{1}{N}}, \phi), \mathcal{Y}^{\mathcal{M}_x}(\iota_{z_p}(A)) \cdot (z_p^{\frac{1}{N}}, v) \rangle = \langle \phi, Y^{M_p^{h_p}}(A, z_p^{\frac{1}{N}}) \cdot v \rangle. \tag{3.9}$$

Then this section $\mathcal{Y}^{\mathcal{M}_x}$ is independent of the choice of special coordinate $z_p^{\frac{1}{N}}$ on each $\mathcal{D}_p^\times$. Furthermore, it is H–equivariant.

Proof. We begin by checking coordinate-independence. Choose a $p \in \nu^{-1}(x)$. Let $w^{\frac{1}{N}}$ be another special coordinate at p. Then there exists a unique $\rho \in \mathrm{Aut}_N \mathcal{O}$ such that $w^{\frac{1}{N}} = \rho(z_p^{\frac{1}{N}})$ (thus, $w = \rho^N(z_p^{\frac{1}{N}})$). We attach to $w^{\frac{1}{N}}$ a section $\widetilde{\mathcal{Y}}_p^{M^\sigma}$ of $\mathcal{V}^*_{\overset{\circ}{C}}|_{\mathcal{D}_p^\times}$ with values in $\mathrm{End}(\mathcal{M}_x)$ by the formula

$$\langle (w^{\frac{1}{N}}, \widetilde{\phi}), \widetilde{\mathcal{Y}}^{\mathcal{M}_x}(\iota_w(\widetilde{A})) \cdot (w^{\frac{1}{N}}, \widetilde{v}) \rangle = \langle \widetilde{\phi}, Y^{M^{h_p}}(\widetilde{A}, w^{\frac{1}{N}}) \cdot \widetilde{v} \rangle.$$

We must show that $\widetilde{\mathcal{Y}}^{\mathcal{M}_x} = \mathcal{Y}^{\mathcal{M}_x}$. We have

$$(z_p^{\frac{1}{N}}, \phi) = (w^{\frac{1}{N}}, \phi \cdot R^{M^{h_p}}(\rho))$$

$$(z_p^{\frac{1}{N}}, v) = (w^{\frac{1}{N}}, R^{M^{h_p}}(\rho)^{-1} \cdot v)$$

As explained in [14], the section $\iota_{z_p}(A)$ of $\mathcal{V}_{\overset{\circ}{C}}$ appears in the coordinate $w = \rho^N(z_p^{\frac{1}{N}})$ as $\iota_w(R^V((\rho^N)_z)^{-1} \cdot A)$. It follows that

$$\iota_{z_p}(A) = \iota_w(R^V((\rho_z^N)^{-1}) \cdot A).$$

Therefore

$$\begin{aligned}&\langle (z_p^{\frac{1}{N}}, \phi), \widetilde{\mathcal{Y}}^{\mathcal{M}_x}(\iota_{z_p}(A) \cdot (z_p^{\frac{1}{N}}, v)\rangle \\ &\qquad = \langle \phi, R^{M^{hp}}(\rho) Y^{M^\sigma}(R^V((\rho^N)_z)^{-1} \cdot A, \rho(z_p^{\frac{1}{N}})) R^{M^{hp}}(\rho)^{-1} v\rangle.\end{aligned}$$

By (3.4), $\widetilde{\mathcal{Y}}^{\mathcal{M}_x} = \mathcal{Y}^{\mathcal{M}_x}$, and we obtain that our section is coordinate-independent.

We now proceed to show that $\mathcal{Y}^{\mathcal{M}_x}$ is H–equivariant. This amounts to checking, for $g \in H$

$$\langle (z_p^{\frac{1}{N}}, \phi), \mathcal{Y}^{\mathcal{M}_x}(g \cdot \iota_{z_p}(A)) \cdot (z_p^{\frac{1}{N}}, v)\rangle = g \cdot \langle (z_p^{\frac{1}{N}}, \phi), \mathcal{Y}^{\mathcal{M}_x}(\iota_{z_p}(A)) \cdot (z_p^{\frac{1}{N}}, v)\rangle$$

where the action on the right is by pullback of functions. The right-hand side is $\langle \phi, Y^{M_p^{hp}}(A, z_p^{\frac{1}{N}} \circ g^{-1}) \cdot v\rangle$. On the left, we have

$$\begin{aligned} g \cdot \iota_{z_p}(A) &= \iota_{z_p \circ g^{-1}}(g \cdot A) \\ (z_p^{\frac{1}{N}}, \phi) &\cong (z_p^{\frac{1}{N}} \circ g^{-1}, \phi \circ T_{g,p,g(p)}^{-1}) \in \mathcal{M}_x^* \\ (z_p^{\frac{1}{N}}, v) &\cong (z_p^{\frac{1}{N}} \circ g^{-1}, T_{g,p,g(p)} \cdot v) \in \mathcal{M}_x \end{aligned}$$

Thus we get

$$\begin{aligned}&\langle (z_p^{\frac{1}{N}}, \phi), \mathcal{Y}^{\mathcal{M}_x}(g \cdot \iota_{z_p}(A) \cdot (z_p^{\frac{1}{N}}, v)\rangle \\ &\qquad = \langle (z_p^{\frac{1}{N}} \circ g^{-1}, \phi \circ T_{g,p,g(p)}^{-1}), \mathcal{Y}^{\mathcal{M}_x}(\iota_{z_p \circ g^{-1}}(g \cdot A)), (z_p^{\frac{1}{N}} \circ g^{-1}, T_{g,p,g(p)} \cdot v)\rangle \\ &\qquad = \langle \phi, T_{g,p,g(p)}^{-1} Y^{M_{g(p)}^{h_{g(p)}}}(g \cdot A, z_p^{\frac{1}{N}} \circ g^{-1}) T_{g,p,g(p)} \cdot v\rangle \\ &\qquad = \langle \phi, Y^{M_p^{hp}}(A, z_p^{\frac{1}{N}} \circ g^{-1}) \cdot v\rangle \\ &\qquad = g \cdot \langle (z_p^{\frac{1}{N}}, \phi), \mathcal{Y}^{\mathcal{M}_x}(\iota_{z_p}(A)) \cdot (z_p^{\frac{1}{N}}, v)\rangle \end{aligned}$$

$\square$

Remark 4. In view of the comments at the beginning of section 3.3.4, $\mathcal{Y}^{\mathcal{M}_x}$ is the pullback under ν of a unique section of $\mathcal{V}^{H,*}_{\overset{\circ}{X}}$ on $\mathcal{D}^{\times}_x$. We will abuse notation by denoting the latter by $\mathcal{Y}^{\mathcal{M}_x}$ as well.

In the case of twisted primary fields, (3.6) implies the following analogue of Proposition 5.3.8 of [14]:

Proposition 3.3.1. *Let $x \in X$, and $\mathcal{M}_x$ a V–module along $\nu^{-1}(x)$. For each $p \in \nu^{-1}(x)$, choose an h_p–special coordinate $z_p^{\frac{1}{N}}$ at p. Define an* $\mathrm{End}(\mathcal{M}_x)$*–valued Δ–differential ϖ on $\coprod_{p\in\nu^{-1}(x)} \mathcal{D}_p^{\times}$ by the formula*

$$\begin{aligned}\langle (z_p^{\frac{1}{N}}, \phi), \varpi \cdot (z_p^{\frac{1}{N}}, v)\rangle &= \langle \phi, Y^{M_p^{h_p}}(A, z_p^{\frac{1}{N}}) \cdot v\rangle (dz_p)^{\Delta} \\ &= N^{\Delta} z_p^{\Delta\frac{(N-1)}{N}} \langle \phi, Y^{M_p^{h_p}}(A, z_p^{\frac{1}{N}}) \cdot v\rangle (dz_p^{\frac{1}{N}})^{\Delta}\end{aligned}$$

Then ϖ is independent of the choice of $z_p^{\frac{1}{N}}$'s.

Recall from Section 5.4.9 of [14] that a primary vector $A \in V$ determines a line subbundle $j_A : \Omega_C^{-\Delta} \hookrightarrow \mathcal{V}_C$, and by dualizing a surjection $j_A^* : \mathcal{V}_C^* \twoheadrightarrow \Omega_C^{\Delta}$. The section ϖ of $\Omega^{\Delta}_{\overset{\circ}{C}}|_{\coprod_{p\in\nu^{-1}(x)} \mathcal{D}_p^{\times}}$ appearing in Proposition 3.3.1 is just the image of the section $\mathcal{Y}^{\mathcal{M}_x}$ under j_A^*.

3.3.5 Dual version

Let $x \in X$, and $\mathcal{M}_x$ a V–module along $\nu^{-1}(x)$. As in the case of ordinary vertex operators (see Section 5.4.8 of [14]), dualizing the construction we obtain a linear map

$$\mathcal{Y}^{\mathcal{M}_x,\vee} : \Gamma(\mathcal{D}_x^{\times}, \mathcal{V}^H_{\overset{\circ}{X}} \otimes \Omega_{\overset{\circ}{X}}) \to \mathrm{End}\, \mathcal{M}_x.$$

Given by

$$s \to \mathrm{Res}_x \langle \mathcal{Y}^{\mathcal{M}_x}, s\rangle \tag{3.10}$$

Moreover, this map factors through the quotient

$$U(\mathcal{V}^H_x) \overset{\mathrm{def}}{=} \Gamma(\mathcal{D}_x^{\times}, \mathcal{V}^H_{\overset{\circ}{X}} \otimes \Omega_{\overset{\circ}{X}}) / \operatorname{Im} \nabla^H,$$

which has a natural Lie algebra structure. The corresponding map $U(\mathcal{V}_x^H) \to \operatorname{End} \mathcal{M}_x$ is a homomorphism of Lie algebras. Note that x does not have to lie in $\overset{\circ}{X}$, but can be *any* point of X.

3.3.6 A sheaf of Lie algebras

Following Section 8.2.5 of [14], let us consider the following complex of sheaves (in Zariski topology) on $\overset{\circ}{X}$:

$$0 \to \mathcal{V}^H_{\overset{\circ}{X}} \xrightarrow{\nabla} \mathcal{V}^H_{\overset{\circ}{X}} \otimes \Omega_{\overset{\circ}{X}} \to 0$$

where $\mathcal{V}^H_{\overset{\circ}{X}} \otimes \Omega_{\overset{\circ}{X}}$ is placed in cohomological degree 0 and $\mathcal{V}^H_{\overset{\circ}{X}}$ is placed in cohomological degree -1 (shifted de Rham complex). Let $h(\mathcal{V}^H_{\overset{\circ}{X}})$ denote the sheaf of the 0th cohomology, assigning to every Zariski open subset $\Sigma \subset \overset{\circ}{X}$ the vector space

$$U_\Sigma(\mathcal{V}^H_{\overset{\circ}{X}}) \overset{\mathrm{def}}{=} \Gamma(\Sigma, \mathcal{V}^H_{\overset{\circ}{X}} \otimes \Omega_{\overset{\circ}{X}}) / \operatorname{Im} \nabla^H$$

One can show as in Chapter 18 of [14] that this is a sheaf of Lie algebras.

According to formula (3.3.5), for any $x \in \Sigma'$, where $\Sigma' \subset X$ is such that $\Sigma' \cap \overset{\circ}{X} = \Sigma$, restriction induces a Lie algebra homomorphism $U_\Sigma(\mathcal{V}^H_{\overset{\circ}{X}}) \to U(\mathcal{V}^H_x)$. We denote the image by $U_\Sigma(\mathcal{V}^H_x)$.

Chapter 4

Conformal Blocks and Twisted Coinvariants

4.1 Conformal Blocks

We use the notation of section 3.3.2. Let $\{x_i\}_{i=1\cdots m}$ be a collection of distinct points of X, which contains all of the branch points of ν. Let $\{\mathcal{M}_{x_i}\}_{i=1\cdots m}$ be a collection of V–modules, such that $\mathcal{M}_{x_i}$ is a module along $\nu^{-1}(x_i)$. Let

$$\mathcal{F} = \bigotimes_{i=1}^{m} \mathcal{M}_{x_i}$$

Given $\phi \in \mathcal{F}^*$, $A_i \in \mathcal{M}_{x_i}$,

$$\langle \phi, A_1 \otimes \cdots \otimes \mathcal{Y}^{\mathcal{M}_{x_i}} \cdot A_i \otimes \cdots \otimes A_m \rangle \tag{4.1}$$

is a section of $\mathcal{V}^{H,*}_{\overset{\circ}{X}}$ on $\mathcal{D}^\times_{x_i}$.

We can now define the generalized space of conformal blocks, extending Definition 9.1.1 of [14]:

Definition 4.1.1. The space of conformal blocks

$$\mathcal{C}_V(X, \{x_i\}, \mathcal{M}_{x_i})_{i=1\cdots m}$$

is by definition the vector space of all linear functionals $\phi \in \mathcal{F}^*$ such that for any $A_i \in \mathcal{M}_{x_i}$, the sections 4.1 can be extended to the *same* section of $\mathcal{V}^{H,*}_{\overset{\circ}{X}}$, regular over $X\backslash\{x_i\}$.

Remark. Observe that in this definition *all* branch points of ν are required to carry module insertions, so that $X\backslash\{x_i\} \subset \overset{\circ}{X}$. $\square$

We can pull back the sections 4.1 by ν to sections of $\mathcal{V}*_{\overset{\circ}{C}}$ on $\coprod_{p\in\nu^{-1}(x_i)} \mathcal{D}_p^\times$ Equivalently, Definition 4.1.1 can be rephrased as follows:

Definition 4.1.2. The space of conformal blocks

$$\mathcal{C}_V(X, \{x_i\}, \mathcal{M}_{x_i})_{i=1\cdots m}$$

is by definition the vector space of all linear functionals $\phi \in \mathcal{F}^*$ such that for any $A_i \in \mathcal{M}_{x_i}$, the pullbacks of the sections 4.1 can be extended to the *same* H–equivariant section of $\mathcal{V}^*_{\overset{\circ}{C}}$, regular over $C\backslash\{\nu^{-1}(x_i)\}$.

4.1.1 Alternative definition

Composing the map (3.10) with the map

$$U_{X\backslash\{x_i\}}(\mathcal{V}^H_{\overset{\circ}{X}}) \to \bigoplus_{i=1}^{m} U(\mathcal{V}^H_{x_i})$$

we obtain an action of the Lie algebra $U_{X\backslash\{x_i\}}(\mathcal{V}^H)$ on $\mathcal{F}$. We will employ the Strong Residue Theorem (see [24]):

Theorem 4.1.1. *Let $\mathcal{E}$ be a vector bundle on a smooth projective curve Z. Let $t_1, \dots, t_n \in Z$ be a set of distinct points. Then a section*

$$\tau \in \bigoplus_{i=1}^{n} \Gamma(\mathcal{D}^\times_{t_i}, \mathcal{E}^*)$$

has the property that

$$\sum_{i=1}^{n} \mathrm{Res}_{t_i}\langle \mu, \tau\rangle = 0, \qquad \forall \mu \in \Gamma(Z - \{t_1, \cdots, t_n\}, \mathcal{E} \otimes \Omega_Z)$$

if and only if τ can be extended to a regular section of $\mathcal{E}$ over $Z - \{t_1, \cdots, t_n\}$ (i.e., $\tau \in \Gamma(Z - \{t_1, \cdots, t_n\}, \mathcal{E}^*)$)

Applying Theorem 4.1.1 to Definition 4.1.1, we obtain that $\phi \in \mathcal{F}^*$ is a conformal block if and only if it vanishes on all elements of the form $s \cdot v,\ s \in U_{X\setminus\{x_i\}}(\mathcal{V}^H_{\overset{\circ}{X}}), v \in \mathcal{F}$. This leads to the following equivalent definition, extending Definition 9.1.2 of [14]:

Definition 4.1.3. The space of *coinvariants* is the vector space

$$\mathcal{H}_V(X, \{x_i\}, \mathcal{M}_{x_i})_{i=1\cdots=}\mathcal{F}/U_{X\setminus\{x_i\}}(\mathcal{V}^H)\cdot\mathcal{F}.$$

The space of *conformal blocks* is its dual: the vector space of $U_{\overset{\circ}{X}}(\mathcal{V}^H_{\overset{\circ}{X}})$–invariant functionals on $\mathcal{F}$

$$\mathcal{C}_V(X, \{x_i\}, \mathcal{M}_{x_i})_{i=1\cdots m} = \mathrm{Hom}_{U_{X\setminus\{x_i\}}(\mathcal{V}^H_{\overset{\circ}{X}})}(\mathcal{F}, \mathbb{C}).$$

4.2 Example: Heisenberg Vertex Algebra

The definition of conformal blocks given in Section 4.1 is quite abstract, and involves a priori all of the fields of the vertex algebra V. When vertex algebras are generated by a finite number of fields the definition of conformal block can be simplified to involve only those generating fields. In this section we illustrate this in the case of the Heisenberg vertex algebra with an order 2 automorphism.

4.2.1 The vertex algebra π and its $\mathbb{Z}/2\mathbb{Z}$–twisted sector

Let $\mathcal{H}$ (resp. $\mathcal{H}^\sigma$) denote the Lie algebra with generators $\{\widetilde{b}_n, \widetilde{K}\}_{n\in\mathbb{Z}}$ (resp. $\{b_n, K\}_{n\in\frac{1}{2}+\mathbb{Z}}$), and commutation relations

$$[\widetilde{b}_n, \widetilde{b}_m] = n\delta_{n,-m}\widetilde{K}$$

(resp. same but with b_n's) where $\widetilde{K}$ (resp. K) is central. Let $\mathcal{H}_+$ (resp. $\mathcal{H}^\sigma_+$) be the subalgebras generated by $\{\widetilde{b}_n\}_{n\geq 0}$ (resp. $\{b_n\}_{n>0}$). For $\lambda \in \mathbb{C}$, let $\widetilde{\mathbb{C}^\lambda}$ denote the 1 dimensional representation of $\mathcal{H}_+ \oplus \mathbb{C}\cdot\widetilde{K}$ on which $\widetilde{b}_n, n > 0$ acts by 0, $\widetilde{b}_0$ acts by λ, and $\widetilde{K}$ acts by the identity. Let

$$\pi^\lambda = \mathrm{Ind}^{\mathcal{H}}_{\mathcal{H}_+\oplus\mathbb{C}\cdot\widetilde{K}}\widetilde{\mathbb{C}^\lambda}.$$

It is well-known that $\pi = \pi^0$ has the structure of a vertex algebra (see for instance Chapter 2 of [14]), generated by the field assignment

$$Y^{\pi}(\widetilde{b}_{-1}|0\rangle, z) = \widetilde{b}(z) = \sum_{n\in\mathbb{Z}} \widetilde{b}_n z^{-n-1}.$$

Let us take $\omega = \frac{1}{2}\widetilde{b}_{-1}^2$ to be the conformal vector of π. With this conformal structure, π has a conformal automorphism σ of order 2, induced from the automorphism of $\mathcal{H}$ which acts by $\widetilde{b}_n \to -\widetilde{b}_n$. All π^{λ} have the structure of conformal π–modules. If $\lambda \in 2\mathbb{Z}$, then π^{λ} is admissible. We write

$$Y^{\pi^{\lambda}}(\widetilde{b}_{-1}|0\rangle, z) = \widetilde{b}^{\lambda}(z) = \sum_{n\in\mathbb{Z}} \widetilde{b}_n z^{-n-1}.$$

Now let $\mathbb{C}$ denote the 1–dimensional representation of $\mathcal{H}_+^{\sigma} \oplus \mathbb{C}\cdot K$ on which K acts by the identity. The $\mathcal{H}^{\sigma}$–module

$$\pi^{\sigma} = \operatorname{Ind}_{\mathcal{H}_+^{\sigma}\oplus\mathbb{C}\cdot K}^{\mathcal{H}^{\sigma}} \mathbb{C}$$

has the structure of an admissible conformal σ–twisted π–module, generated (in the sense of [23]) by the field assignment

$$Y^{\pi^{\sigma}}(\widetilde{b}_{-1}|0\rangle, z) = b(z^{\frac{1}{2}}) = \sum_{n\in\frac{1}{2}+\mathbb{Z}} b_n z^{-n-1}.$$

The twisted vertex operator assigned to an arbitrary vector $v \in \pi$ is given as follows (see [16, 20, 6]). Let

$$W^{\pi^{\sigma}}(\widetilde{b}_{n_1}\ldots\widetilde{b}_{n_k}|0\rangle, z) = \frac{1}{(-n_1-1)!}\cdots\frac{1}{(-n_k-1)!}{:}\partial_z^{-n_1-1}b(z)\ldots\partial_z^{-n_k-1}b(z){:}$$

and set

$$\Delta_z = \sum_{m,n\geq 0} c_{mn}\widetilde{b}_m\widetilde{b}_n z^{-m-n}$$

where the constants c_{mn} are determined by the formula

$$\sum_{m,n\geq 0} c_{mn}x^m y^n = -\log\left(\frac{(1+x)^{1/2}+(1+y)^{1/2}}{2}\right)$$

Then for any $v \in \pi$ we have

$$Y^{\pi^{\sigma}}(v, z) = W^{\pi^{\sigma}}(\exp\Delta_z\cdot v, z). \tag{4.2}$$

4.2.2 The Lie algebra $\mathcal{H}_{\text{out}}$

Using the notation of earlier sections, we now restrict to the case where $V = \pi$, $H \cong \langle \sigma_C \rangle$, where σ_C has order 2, and $\nu : C \to X$ has degree 2. This is for example the case when C is hyperelliptic, σ_C is the hyperelliptic involution, and $X = \mathbb{CP}^1$. The vector bundles $\mathcal{V}_{\overset{\circ}{C}}, \mathcal{V}^H_{\overset{\circ}{X}}$ will be denoted $\Pi_{\overset{\circ}{C}}, \Pi^{\sigma}_{\overset{\circ}{X}}$ respectively.

We begin by discussing how to construct π–modules along $\nu^{-1}(x)$ for $x \in X$. If $\nu^{-1}(x)$ consists of a single point, we get a module along $\nu^{-1}(x)$ by starting with π^{σ}, and proceeding as in example 3.3.2. Denote the corresponding module along $\nu^{-1}(x)$ by π^{σ}_x. If $\nu^{-1}(x)$ consists of two points, we can proceed as follows. Pick $\lambda \in 2\mathbb{Z}$, and a point $p \in \nu^{-1}(x)$. Define $T_{\sigma_\pi, p, \sigma_C(p)} : \pi^{\lambda}_p \to \pi^{-\lambda}_{\sigma_C(p)}$ by

$$T_{\sigma_\pi, p, \sigma_C(p)}(\widetilde{b_{i_1}} \cdots \widetilde{b_{i_k}} |0\rangle_\lambda) \to (-1)^k \widetilde{b_{i_1}} \cdots \widetilde{b_{i_k}} |0\rangle_{-\lambda}$$

It is easy to see that

$$T^{-1}_{\sigma_\pi, p, \sigma_C(p)} Y^{\pi^{-\lambda}_{\sigma_C(p)}}(\sigma_\pi(A), z) T_{\sigma_\pi, p, \sigma_C(p)} = Y^{\pi^{\lambda}_p}(A, z),$$

and so the construction of Definition 3.3.3 applies. Denote the corresponding module along $\nu^{-1}(x)$ by $\pi^{\lambda,p}_x$

Remark 5. Note that this construction requires a choice of λ and a point in the fiber $\nu^{-1}(x)$.

Let $\{x_i\}_{i=1\cdots m}$ be a collection of points of X containing all of the branch points of ν, and $\{\pi_{x_i}\}$ a collection of π–modules along $\nu^{-1}(x_i)$, where $\pi_{x_i} \cong \pi^{\sigma}_{x_i}$ or $\pi_{x_i} \cong \pi^{\lambda_i, p_i}_{x_i}$ depending on whether $\nu^{-1}(x_i)$ consists of one or two points. The vector $\widetilde{b}_{-1}|0\rangle \in \pi$ is primary and has conformal weight 1. Applying Proposition 3.3.1, for each x_i, we obtain an $\mathrm{End}(\pi_{x_i})$–valued 1–form ϖ_i on $\coprod_{p \in \nu^{-1}(x_i)} \mathcal{D}^{\times}_p$ Let

$$\mathcal{F} = \bigotimes_{i=1}^{m} \pi_{x_i}$$

and let $\mathcal{H}_{\text{out}}(C_{aff})$ be the abelian Lie algebra $\mathbb{C}[C \backslash \{\nu^{-1}(x_i)\}]$ of regular functions on $C \backslash \{\nu^{-1}(x_i)\}$. If $A_i \in \pi_{x_i}$, then $f \in \mathcal{H}_{\text{out}}(C_{aff})$ acts on $\mathcal{F}$ by

$$f \cdot (A_1 \otimes \cdots \otimes A_m) = \sum_{i=1}^{m} \sum_{p \in \nu^{-1}(x_i)} A_1 \otimes \cdots \otimes (\mathrm{Res}_p f \varpi_i) A_i \otimes \cdots \otimes A_m \qquad (4.3)$$

We will say that a meromorphic function f on C is *even* if $\sigma_C^*(f) = f$ and *odd* if $\sigma_C^*(f) = -f$. If f is even, then $\sum_{p\in\nu^{-1}(x_i)} \mathrm{Res}_p\, f\varpi_i = 0$, so only odd functions act non-trivially. Let $\mathcal{H}_{\mathrm{out}}(C_{aff})^o$ denote the space of odd elements in $\mathbb{C}[C\backslash\{\nu^{-1}(x_i)\}]$.

4.2.3 Coinvariants and conformal blocks

Now we give an simpler, alternative definition of the spaces of coinvariants and conformal blocks for (twisted) π–modules, extending Definition 8.1.7 in [14].

Definition 4.2.1. The space of *coinvariants* associated to $(X, \{x_i\}, \{\pi_{x_i}\})$ is the vector space

$$\widetilde{\mathcal{H}}_\pi(X, \{x_i\}, \pi_{x_i})_{i=1\cdots m} = \mathcal{F}/\mathcal{H}_{\mathrm{out}}(C_{aff})^o \cdot \mathcal{F}$$

The space of *conformal blocks* associated to $(X, \{x_i\}, \{\pi_{x_i}\})$ is the vector space

$$\widetilde{\mathcal{C}}_\pi(X, \{x_i\}, \pi_{x_i})_{i=1\cdots m} = \mathrm{Hom}_{\mathcal{H}_{\mathrm{out}}(C_{aff})^o}(\mathcal{F}, \mathbb{C})$$

of $\mathcal{H}_{\mathrm{out}}(C_{aff})^o$–invariant functionals on $\mathcal{F}$.

Remark. The discussion at the end of Section 4.2.2 implies that these definitions remain the same if we replace $\mathcal{H}_{\mathrm{out}}(C_{aff})^o$ by $\mathcal{H}_{\mathrm{out}}(C_{aff})$. □

We can ask how the space of conformal blocks changes under the addition of points. Suppose then that to our collection of points $\{x_i\}_{i=1\cdots m}$ we add x_{m+1}. Since $\{x_i\}_{i=1\cdots m}$ contains all the branch points of ν, $\nu^{-1}(x_{m+1})$ consists of two points. Set $\pi_{x_{m+1}} = \pi^{0,p}_{x_{m+1}}$, $p \in \nu^{-1}(x_{m+1})$. Observe that in the case of the vacuum representation, $\pi^{0,p}_{x_{m+1}} = \pi^{0,\sigma_C(p)}_{x_{m+1}}$, so there is no choice of point in the fiber. There exists a natural map:

$$\widetilde{\mathcal{C}}_\pi(X, \{x_i\}, \pi_{x_i})_{i=1\cdots m+1} \to \widetilde{\mathcal{C}}_\pi(X, \{x_i\}, \pi_{x_i})_{i=1\cdots m} \tag{4.4}$$

given by

$$\phi \to \phi|_{\bigotimes_{i=1}^m \pi_{x_i}\otimes|0\rangle}, \tag{4.5}$$

i.e., we restrict the functional ϕ to the vacuum vector in $\pi_{x_{m+1}}$. The following lemma, analogous to Proposition 8.3.2 in [14], will be proved in Section 4.2.5.

Lemma 4.2.1. *The map* (4.5) *is an isomorphism.*

4.2.4 Equivalence of Definitions 4.1.1 and 4.2.1

We now have two seemingly different definitions of the space of conformal blocks: the general Definition 4.1.1 and Definition 4.2.1, which is specific to the case $V = \pi$. We will show that these two definitions agree. Applying the Strong Residue Theorem 4.1.1 to our collection of $\mathrm{End}(\mathcal{F})$–valued one-forms, we obtain

Corollary 4.2.1. *A functional ϕ is a conformal block if and only if $\forall A_i \in \pi_{x_i}$, the one-forms*

$$\langle \phi, A_1 \otimes \cdots \otimes (\varpi_i \cdot A_i) \otimes \cdots \otimes A_m \rangle \in \Gamma(\coprod_{p \in \nu^{-1}(x_i)} \mathcal{D}_p^{\times}, \Omega_{\overset{\circ}{C}}) \tag{4.6}$$

can be extended to a single one-form ϖ_ϕ on $C \backslash \{\nu^{-1}(x_i)\}$.

Remark. σ_C acts on the space of holomorphic one-forms. It is clear that ϖ_ϕ is odd under this action. $\square$

We are now ready to prove the equivalence of the two definitions of conformal blocks. The proof is a generalization of the proof of Theorem 8.3.3 of [14].

Theorem 4.2.1. *Let ϕ be a linear functional on $\mathcal{F}$ such that $\forall A_i \in \pi_{x_i}$, the one-forms (4.6) can be extended to a single, odd, regular one–form ϖ_ϕ on $C \backslash \{\nu^{-1}(x_i)\}$. Then the sections*

$$\langle \phi, A_1 \otimes \cdots \otimes (\mathcal{Y}^{\pi_{x_i}} \cdot A_i) \otimes \cdots \otimes A_m \rangle \in \Gamma(\coprod_{p \in \nu^{-1}(x_i)} \mathcal{D}_p^{\times}, \Pi_C^*) \tag{4.7}$$

can be extended to a single, equivariant, regular section of Π_C^* on $C \backslash \{\nu^{-1}(x_i)\}$, and vice versa.

Proof. From the discussion following Theorem 3.3.1, there exists a map

$$\Pi^*_{\overset{\circ}{C}} \to \Omega_{\overset{\circ}{C}}$$

such that the one-forms (4.6) are simply the projections of the sections (4.1) on $\coprod_{p \in \nu^{-1}(x_i)} \mathcal{D}_p^{\times}$ Thus if the (4.1) extend to $C \backslash \{\nu^{-1}(x_i)\}$, so will (4.6). Denote $C \backslash \{\nu^{-1}(x_i)\}$ by C_{aff}.

Let $\hat{C}^2$ denote $C^2_{aff}\backslash\Xi$, where Ξ is the divisor in C^2_{aff} consisting of pairs $(x,y) \in C^2_{aff}$ satisfying $\sigma^k_C(x) = \sigma^l_C(y)$ for some integers k,l. Let $r_1 : \hat{C}^2 \to C_{aff}$ denote the projection on the first factor, whose fiber over $q \in C_{aff}$ is

$$C_{aff}\backslash\mathbf{O}_q$$

where $\mathbf{O}_q$ denotes the H–orbit of q. Let

$$\widetilde{\mathcal{O}} = (r_1)_*\mathcal{O}_{\hat{C}^2}.$$

This is a quasi-coherent sheaf on C_{aff} whose fiber at $q \in C_{aff}$ is $\mathbb{C}[C_{aff}\backslash\mathbf{O}_q]$. Let $\mathcal{G}$ denote the sheaf $\mathcal{F} \otimes \Pi_C|_{C_{aff}}$ on C_{aff}, where we treat $\mathcal{F}$ as a constant sheaf. For $q \in C_{aff}$, the fiber $\mathcal{G}_q$ is $\mathcal{F} \otimes (\Pi_C)_q \cong \mathcal{F} \otimes \pi_{\nu(q)}$, where by $\pi_{\nu(q)}$ we mean the module along $\nu^{-1}(\nu(q))$ constructed out of the vacuum π. The sheaf $\widetilde{\mathcal{O}}$ acts on $\mathcal{G}$ in such a way that fibrewise we obtain the action (4.3) of $\mathbb{C}[C_{aff}\backslash\mathbf{O}_q]$ on

$$\mathcal{F} \otimes \Pi_{\nu(q)} = \bigotimes_{i=1}^{m} \pi_{x_i} \otimes \pi_{\nu(q)}$$

Introduce the sheaf of homomorphisms

$$\widetilde{\mathcal{C}} = \mathrm{Hom}_{\widetilde{\mathcal{O}}}(\mathcal{G}, \mathcal{O}_{C_{aff}})$$

whose fiber at $q \in C_{aff}$ is the space of conformal blocks

$$\widetilde{\mathcal{C}}_\pi(X, \{x_i, \nu(q)\}, \pi_{x_i}, \pi_{\nu(q)})_{i=1\cdots m}.$$

Lemma 4.2.1 identifies the fibers of $\widetilde{\mathcal{C}}$ with $\widetilde{\mathcal{C}}_\pi(X, \{x_i\}, \pi_{x_i})_{i=1\cdots r}s$, thus providing a canonical trivialization of $\widetilde{\mathcal{C}}$. Thus, given

$$\phi \in \widetilde{\mathcal{C}}_\pi(X, \{x_i\}, \pi_{x_i})_{i=1\cdots m}$$

we obtain a section $\widetilde{\phi}$ of $\widetilde{\mathcal{C}}$ on C_{aff}. Let $\widetilde{\phi}_q$ denote the value of this section at $q \in C$. Let $A_i \in \pi_{x_i}$ and

$$v = \otimes_{i=1}^m A_i \in \mathcal{F}$$

Contracting $\widetilde{\phi}$ with $v \in \mathcal{F}$, we obtain a section $\widetilde{\phi}^v$ of Π^*_C. By construction, the value of $\widetilde{\phi}^v$ on $B \in (\Pi_C)_q = \pi_{\nu(q)}$ is $\widetilde{\phi}_q(v \otimes B)$. We wish to show that the section $\widetilde{\phi}^v$ is the analytic continuation of the sections (4.1). We begin with the following lemma.

Lemma 4.2.2. *There exists a regular one-form on $C_{aff}\backslash \mathbf{O}_q$ whose restriction to $\coprod_{p\in\nu^{-1}(x_i)} \mathcal{D}_p^\times$ is*

$$\langle \widetilde{\phi}_q, A_1 \otimes \cdots \otimes \varpi_i \cdot A_i \otimes \cdots A_m \otimes B\rangle,$$

and its restriction to $\coprod_{p\in\nu^{-1}(\nu(q))} \mathcal{D}_p^\times$ is

$$\langle \widetilde{\phi}_q, A_1 \otimes \cdots \otimes A_m \otimes \varpi_q \cdot B\rangle$$

Proof. Since $\widetilde{\phi}_q$ is a conformal block, we obtain, using the definition of the action of $\mathbb{C}[C_{aff}\backslash \mathbf{O}_q]$ on $\mathcal{F} \otimes \pi_{\nu(q)}$,

$$\begin{aligned} 0 = \langle \widetilde{\phi}_q, f \cdot (v \otimes B)\rangle = &\sum_{i=1\cdots m} \sum_{p\in\nu^{-1}(x_i)} \mathrm{Res}_p \langle \widetilde{\phi}_q, A_1 \otimes \cdots \otimes (f\varpi_i) \cdot A_i \otimes \cdots \otimes B\rangle \\ &+ \sum_{p=(q,\sigma_C(q))} \mathrm{Res}_p \langle \widetilde{\phi}_q, A_1 \otimes \cdots \otimes A_m \otimes \cdots \otimes (f\varpi_q) \cdot B\rangle \end{aligned}$$

Thus, the strong residue theorem implies that there exists a regular one-form on $C_{aff}\backslash \mathbf{O}_q$ having the desired properties. □

Remark 6. If $x \in X, p \in \nu^{-1}(x)$, and z is a formal coordinate at p, then the endomorphism-valued one-form ϖ on $\mathcal{D}_p^\times$ has the expression

$$\varpi = N z^{\frac{N-1}{N}} b(z^{\frac{1}{N}}) dz^{\frac{1}{N}}$$

where

$$b(z^{\frac{1}{N}}) = \sum_{n\in\frac{1}{N}+\mathbb{Z}} b_n z^{-n-1}$$

and where N is the number of points in $\nu^{-1}(x)$.

Now we prove Theorem 4.2.1. Suppose that q is near $p \in \nu^{-1}(x_i)$. Choose a small analytic neighborhood U of p with special coordinate $z_i^{\frac{1}{N}}$ centered on p, such that $q \in U_i$. If ν is unramified, $N = 1$, and $z_i^{\frac{1}{N}}$ is any coordinate centered at p, otherwise $N = 2$ and $z_i^{\frac{1}{2}}$ is a σ_C–special coordinate. Since $q \neq p$, $w = z_i - z_i(q)$ is a coordinate centered at q, in some neighborhood W of q. Near q, we can trivialize

$$\pi \times W \cong \Pi_C|_W$$

via

$$(B, q) \to ((B, z_i - z_i(q)), q).$$

It remains to prove the following:

Lemma 4.2.3. *$\forall B \in \pi$,*

$$\begin{aligned}\langle \widetilde{\phi}_q, v \otimes B\rangle &= \langle \widetilde{\phi}_q, A_1 \otimes \cdots \otimes Y^{\pi_{x_i}}(B, q) \cdot A_i \otimes \cdots \otimes A_m \otimes |0\rangle\rangle \\ &= \langle \phi, A_1 \otimes \cdots \otimes Y^{\pi_{x_i}}(B, q) \cdot A_i \otimes \cdots \otimes A_m\rangle.\end{aligned}$$

Proof. The second equality follows from Lemma 4.2.1. The first equality is proved by induction. It obviously holds for $B = |0\rangle$. Now, denote by $\pi^{(r)}$ the subspace of π spanned by all monomials of the form $\widetilde{b_{i_1}} \cdots \widetilde{b}_{i_k}|0\rangle$, where $k \leq r$. Suppose that we have proved Lemma 4.2.3 for all $B \in \pi^{(r)}$. The inductive step is to prove the equality for elements of the form $B' = \widetilde{b_n} \cdot B$. By our inductive hypothesis, we know that if $B \in \pi^{(r)}$, then

$$\begin{aligned}&\langle \widetilde{\phi}_q, A_1 \otimes \cdots \otimes (N z_i^{\frac{N-1}{N}} b(z_i^{\frac{1}{N}})) \cdot A_i \otimes \cdots \otimes B\rangle dz_i^{\frac{1}{N}} \\ &\qquad = \langle \phi, A_1 \otimes \cdots \otimes N Y^{\pi_{x_i}}(B, q) z^{\frac{N-1}{N}} b(z_i^{\frac{1}{N}}) \cdot A_i \otimes \cdots \otimes A_m\rangle dz_i^{\frac{1}{N}}.\end{aligned}$$

According to Lemma 4.2.2 above, we also have

$$\langle \widetilde{\phi}_q, A_1 \otimes \cdots \otimes (N z_i^{\frac{N-1}{N}} b(z_i^{\frac{1}{N}})) \cdot A_i \otimes \cdots \otimes A_m \otimes B\rangle dz_i^{\frac{1}{N}} = \langle \widetilde{\phi}_q, A_1 \otimes \cdots \otimes A_m \otimes \widetilde{b}(w) \cdot B\rangle dw.$$

Using locality and associativity, we obtain

$$\begin{aligned}&\langle \phi, A_1 \otimes \cdots \otimes Y^{\pi_{x_i}}(C, q) b(z_i^{\frac{1}{N}}) \cdot A_i \otimes \cdots \otimes A_m\rangle N z^{\frac{N-1}{N}} dz_i^{\frac{1}{N}} \\ &\qquad = \langle \phi, A_1 \otimes \cdots \otimes b(z_i^{\frac{1}{N}}) Y^{\pi_{x_i}}(C, q) \cdot A_i \otimes \cdots \otimes A_m\rangle N z^{\frac{N-1}{N}} dz_i^{\frac{1}{N}} \\ &\qquad = \langle \phi, A_1 \otimes \cdots \otimes Y^{\pi_{x_i}}(\widetilde{b}(z_i - z_i(q)) \cdot B, q) \cdot A_i \otimes \cdots \otimes A_m\rangle N z^{\frac{N-1}{N}} dz_i^{\frac{1}{N}} \\ &\qquad = \langle \phi, A_1 \otimes \cdots \otimes Y^{\pi_{x_i}}(\widetilde{b}(w) \cdot B, q) \cdot A_i \otimes \cdots \otimes A_m\rangle N z^{\frac{N-1}{N}} dz_i^{\frac{1}{N}} \\ &\qquad = \langle \phi, A_1 \otimes \cdots \otimes Y^{\pi_{x_i}}(\widetilde{b}(w) \cdot B, q) \cdot A_i \otimes \cdots \otimes A_m\rangle dw,\end{aligned}$$

where the last step holds because $Nz^{\frac{N-1}{N}}dz_i^{\frac{1}{N}} = dw$. Combining these relations, we obtain

$$\langle \widetilde{\phi}_q, A_1 \otimes \cdots \otimes A_m \otimes \widetilde{b}(w) \cdot B \rangle dw = \langle \phi, A_1 \otimes \cdots \otimes Y^{\pi_{x_i}}(\widetilde{b}(w) \cdot B, q) \cdot A_i \otimes \cdots \otimes A_m \rangle dw.$$

Multiplying both sides by w^n and taking residues, we find that

$$\langle \widetilde{\phi}_q, A_1 \otimes \cdots \otimes A_m \otimes \widetilde{b}_n \cdot B \rangle = \langle \phi, A_1 \otimes \cdots \otimes Y^{\pi_{x_i}}(\widetilde{b}_n \cdot B, q) \cdot A_i \otimes \cdots \otimes A_m \rangle.$$

Equivariance of the sections 4.7 follows from the fact that they are equivariant on all $\mathcal{D}_p^\times$ where p is a fixed point of σ_C. This completes the proof of Theorem 4.2.1.

□

4.2.5 Proof of Lemma 4.2.1

We start with the following fact.

Lemma 4.2.4. *Let $p \in \nu^{-1}(x_{m+1})$. For every principal part f_- at p, there exists an odd function $f \in \mathbb{C}[C_{aff} \backslash \mathbf{O}_p]$ whose principal part at p is f_-.*

Proof. Let D be an effective divisor symmetric under the action of σ_C (i.e., if $D = \Sigma c_q \cdot q$, then $c_q = c_{\sigma_C(q)}$), supported on $\{\nu^{-1}(x_i)\}_{i=1\cdots m}$. Denote the canonical divisor of C by K_C. For $\deg(D) > \deg(K_C)$, the Riemann-Roch theorem implies that

$$\dim \mathcal{L}(D) = \deg(D) + 1 - g_C.$$

It follows that

$$Q_{n+1} = \mathcal{L}(D + (n+1) \cdot p + (n+1)\sigma_C(p)) / \mathcal{L}(D + n \cdot p + n \cdot \sigma_C(p))$$

for $n \geq 0$ is two-dimensional. Furthermore, Q_{n+1} carries an action of σ_C. Suppose now that Q_{n+1} is spanned by the images of two even functions f_1, f_2. Since f_i are even, they have poles of the same order at both p and $\sigma_C(p)$, and so will any linear combination. But this contradicts the fact that

$$\mathcal{L}(D + (n+1)p + n\sigma_C(p)) / \mathcal{L}(D + n \cdot p + n \cdot \sigma_C(p))$$

is one-dimensional. It follows that for each $n \geq 0$, Q_{n+1} contains an odd function. □

Now we prove the statement equivalent to Lemma 4.2.1 that the corresponding spaces of coinvariants

$$\widetilde{\mathcal{H}}_\pi(X,\{x_i\},\pi_{x_i})_{i=1\cdots m} \quad \text{and} \quad \widetilde{\mathcal{H}}_\pi(X,\{x_i\},\pi_{x_i})_{i=1\cdots m+1}$$

are isomorphic. Recall that $\pi_{x_{m+1}}$ here is the vacuum module $\pi^{0,p}_{x_{m+1}}$. Let $C'_{aff} = C_{aff}\backslash \mathbf{O}_p$, and $\mathcal{H}_{\text{out}}(C'_{aff}) = \mathbb{C}[C'_{aff}]$. The space

$$\widetilde{\mathcal{H}}_\pi(X,\{x_i\},\pi_{x_i})_{i=1\cdots m} \qquad (\text{resp.}\, \widetilde{\mathcal{H}}_\pi(X,\{x_i\},\pi_{x_i})_{i=1\cdots m+1}$$

is identified with the 0th homology of the Lie algebra $\mathcal{H}_{\text{out}}(C_{aff})^o$ (resp. $\mathcal{H}_{\text{out}}(C'_{aff})^o$) with coefficients in $\mathcal{F}$ (resp. $\mathcal{F}\otimes\pi_{x_{m+1}}$). Lemma 4.2.4 implies that the sequence

$$0 \to \mathcal{H}_{\text{out}}(C_{aff})^o \to \mathcal{H}_{\text{out}}(C'_{aff})^o \xrightarrow{\mu} w_{s+1}^{-1}\mathbb{C}[w_{s+1}^{-1}] \to 0 \tag{4.8}$$

is exact, where μ is the map that attaches to a function its principal part at p. The homology of $\mathcal{H}_{\text{out}}(C'_{aff})^o$ with coefficients in $\mathcal{F}\otimes\pi_{x_{m+1}}$ is computed using the Chevalley complex

$$C^\bullet = \mathcal{F}\otimes\pi_{x_{m+1}}\otimes\bigwedge{}^\bullet(\mathcal{H}_{\text{out}}(C'_{aff})^o)$$

with the differential $d: C^i \to C^{i-1}$ given by the formula

$$d = \sum_i f_i\otimes\psi_i^*$$

where $\{f_i\}$ is a basis in $\mathcal{H}_{\text{out}}(C'_{aff})^o$ and $\{\psi^*\}$ is the dual basis of $(\mathcal{H}_{\text{out}}(C'_{aff})^o)^*$ acting on $\bigwedge^\bullet(\mathcal{H}_{\text{out}}(C'_{aff})^o)$ by contraction.

Choose pull-backs $z_n, n<0$, of $w_{s+1}^n, n<0$, in $\mathcal{H}_{\text{out}}(C'_{aff})^o$ under μ. Because of the exactness of the sequence (4.8), we can choose a basis $\{f_i\}$ in $\mathcal{H}_{\text{out}}(C'_{aff})^o$ which is a union of $\{z_n\}_{n<0}$, and a basis of $\mathcal{H}_{\text{out}}(C_{aff})^o$. In this basis we may decompose

$$d = d_{C_{aff}} + \sum z_n\otimes\phi_n^*,$$

where $d_{C_{aff}}$ is the differential for $\mathcal{H}_{\text{out}}(C_{aff})^o$, and ϕ_n^* denotes the element of the dual basis to $\{f_i\}$ corresponding to z_n.

We need to show that the homologies of this complex are isomorphic to the homologies of the complex $\mathcal{F}\otimes\bigwedge^\bullet(\mathcal{H}_{\text{out}}(C_{aff})^o)$. Introduce an increasing filtration on

$\pi_{x_{m+1}}$, letting $\pi^{(r)}_{m+1}$ be the span of all monomials of order less than or equal to m in $\widetilde{b}_n, n < 0$. Now introduce a filtration $\{F_i\}$ on the Chevalley complex $C^\bullet$ by setting

$$F_i = \mathrm{span}\{v \otimes B \otimes D | v \in \mathcal{F}, B \in \pi^{(m)}_{x_{m+1}}, D \in \bigwedge{}^{i-m}(\mathcal{H}_{\mathrm{out}}(C'_{aff})^o)\}.$$

Our differential preserves this filtration.

Consider now the spectral sequence associated to the filtered complex $C^\bullet$. The zeroth term E^0 is the associated graded space of the Chevalley complex, isomorphic to

$$(\pi_{s+1} \otimes \bigwedge{}^\bullet(\phi^*_n)_{n<0}) \otimes (\mathcal{F} \otimes \bigwedge{}^\bullet(\mathcal{H}_{\mathrm{out}}(C_{aff})^o)).$$

The zeroth differential acts along the first factor of the above decomposition, and is given by the formula

$$d^0 = \sum_{n<0} b_n \otimes \phi^*_n,$$

because on the graded module the operator z_n acts as $b_n, n < 0$. But π_{s+1} is isomorphic to the symmetric algebra with generators $b_n, n < 0$, and our differential is simply the Koszul differential for this symmetric algebra. It is well-known that the zeroth homology of this complex is isomorphic to $\mathbb{C}$, and all other homologies vanish. Therefore all positive homologies of d^0 vanish, while the zeroth homology is $\mathcal{F} \otimes \bigwedge^\bullet(\mathcal{H}_{\mathrm{out}}(C_{aff}))$. Hence, the E^1 term coincides as a vector space with the Chevalley complex of the homology of $\mathcal{H}_{\mathrm{out}}(C_{aff})^o$ with coefficients in $\mathcal{F}$. Also, the E^1 differential coincides with $d_{C_{aff}}$, which is the corresponding Chevalley differential. We thus obtain the desired isomorphism

$$H_i(\mathcal{H}_{\mathrm{out}}(C'_{aff})^o, \mathcal{F} \otimes \pi_{m+1}) \cong H_i(\mathcal{H}_{\mathrm{out}}(C_{aff})^o, \mathcal{F}).$$

Chapter 5

Internal Heisenberg Symmetries

In this chapter we extend the discussion of chapters 3, 4 to the case where the vertex algebra V possesses internal Heisenberg symmetries. This means that we are given a conformal vertex algebra V, a conformal automorphism σ_V of order 2, and a $\mathbb{Z}/2\mathbb{Z}$–equivariant embedding of conformal vertex algebras $\iota : \pi \mapsto V$. In this case, the $\operatorname{Aut}\mathcal{O} \times \mathbb{Z}/2\mathbb{Z}$ action on V (by exponentiation of the positive half of the Virasoro algebra and σ_π), is augmented to an action of $(\operatorname{Aut}\mathcal{O} \times \mathbb{Z}/2\mathbb{Z}) \ltimes \mathcal{O}^*$.

We can summarize the results of chapter 3 by saying that the operation Y^M transforms covariantly under the action of the group $\operatorname{Aut}\mathcal{O} \times H$. From this perspective, the current chapter is devoted to proving the analogous result for the larger group $(\operatorname{Aut}\mathcal{O} \times H) \ltimes \mathcal{O}^*$ when $H = \mathbb{Z}/2\mathbb{Z}$.

5.1 Recollections from [14]

We follow the notation of section 4.2. Let $\iota : \pi \mapsto V$ be an embedding of conformal vertex algebras. The Fourier coefficients of the field

$$\widetilde{b}(z) = Y^V(\iota(\widetilde{b}_{-1}|0\rangle), z) = \sum_{n\in\mathbb{Z}} \widetilde{b}_n z^{-n-1}$$

generate an action of the Heisenberg algebra $\mathcal{H}$ on V. If $\widetilde{b}_0$ has integer eigenvalues, then the action of $\mathcal{H}_+$ can be exponentiated to an action of the group $\mathcal{O}^*$. The

Virasoro action coming from the Fourier coefficients of

$$T(z) = Y^V(\iota(\omega), z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

normalizes the action of $\mathcal{H}$, and so we obtain an action of $\operatorname{Aut}\mathcal{O} \ltimes \mathcal{O}^*$ on V. Given a torsor S for this group, we can form the twist

$$\mathcal{V}^S = S \underset{\operatorname{Aut}\mathcal{O}\ltimes\mathcal{O}^*}{\times} V$$

A natural $\operatorname{Aut}\mathcal{O}$ torsor is $\mathcal{A}ut(\mathcal{D}_x)$, the torsor of local coordinates at $x \in X$, where X is a curve. Twisting V by Aut_X (whose fiber at x is $\mathcal{A}ut(\mathcal{D}_x)$), allows us to give a coordinate-free description of the vertex operator. Torsors for the larger group $\operatorname{Aut}\mathcal{O} \ltimes \mathcal{O}^*$ are constructed from line bundles. Given a line bundle $\mathcal{L}$ on X and a point $x \in X$, let $\widehat{\mathcal{L}}_x$ be the set of pairs (z, s) where z is a formal coordinate at x, and s is a trivialization of $\mathcal{L}|_{\mathcal{D}_x}$. $\operatorname{Aut}\mathcal{O}$ acts by coordinate changes as before, and for a fixed coordinate, the different trivializations of $\mathcal{L}|_{\mathcal{D}_x}$ are an $\mathcal{O}^*$–torsor. Thus, $\widehat{\mathcal{L}}_x$ is an $\operatorname{Aut}\mathcal{O} \ltimes \mathcal{O}^*$–torsor. Given a curve X, the pointwise torsors $\forall x \in X$ assemble into an $\operatorname{Aut}\mathcal{O} \ltimes \mathcal{O}^*$ principal bundle $\widehat{\mathcal{L}}$. Let

$$\mathcal{V}^{\mathcal{L}} = \widehat{\mathcal{L}} \underset{\operatorname{Aut}\mathcal{O}\ltimes\mathcal{O}^*}{\times} V$$

and for a V–module M, let

$$\mathcal{M}_x^{\mathcal{L}} = \widehat{\mathcal{L}}_x \underset{\operatorname{Aut}\mathcal{O}\ltimes\mathcal{O}^*}{\times} M$$

We have the following

Theorem 5.1.1. *Given a line bundle $\mathcal{L}$ on X, and $x \in X$, the vertex operation Y^M gives rise to a canonical* $\operatorname{End}(\mathcal{M}_x^{\mathcal{L}})$*–valued section $\mathcal{Y}^{\mathcal{M}_x^{\mathcal{L}}}$ of $\mathcal{V}^{\mathcal{L},*}$ on $\mathcal{D}_x^\times$.*

In this chapter, we obtain a twisted analogue of Theorem 5.1.1, and define a notion of conformal blocks for twisted modules with augmented Heisenberg symmetry.

5.2 Torsors and Twists

We follow the notation of section 4.2. Let V be a conformal vertex algebra, and σ_V a conformal automorphism of V of order 2. We suppose that there is an embedding of conformal vertex algebras $\iota : \pi \mapsto V$ equivariant with respect to the $\mathbb{Z}/2\mathbb{Z}$ action (by σ_π on π, and σ_V on V). Let M^σ be an admissible σ_V–twisted V–module. The Fourier coefficients of the field $Y^{M^\sigma}(\iota(\widetilde{b}_{-1}|0\rangle), z^{\frac{1}{2}})$ generate an action of the twisted Heisenberg Lie algebra $\mathcal{H}^\sigma$ on M^σ.

5.2.1 The Group $\mathcal{O}^*_{odd}$

Definition 5.2.1. Let $\mathcal{O}^*_{odd}$ denote the group-scheme of infinite type (resp. Ind group-scheme) whose R–points, for a commutative $\mathbb{C}$–algebra R, are those elements $\gamma(t^{\frac{1}{2}}) \in R[[t^{\frac{1}{2}}]]$ such that $\gamma(t^{\frac{1}{2}}) \cdot \gamma(-t^{\frac{1}{2}}) = 1$.

Throughout this chapter, by $\mathcal{O}^*_{odd}$ we will understand the $\mathbb{C}$–points of this group-scheme. We have a canonical isomorphism

$$Lie(\mathcal{O}^*_{odd}) \cong \mathcal{H}^\sigma_+$$

The action of $\mathcal{H}^\sigma_+$ on M^σ is locally nilpotent, and so can be exponentiated to an action of $\mathcal{O}^*_{odd}$.

We know from Chapter 3 that M^σ also carries an action of $\mathrm{Aut}_2\,\mathcal{O}$, generated by the Fourier coefficients of $Y^{M^\sigma}(\iota(\omega), z^{\frac{1}{2}})$. Furthermore, it follows from the OPE between the fields $Y^{M^\sigma}(\iota(\omega), z^{\frac{1}{2}}), Y^{M^\sigma}(\iota(\widetilde{b}_{-1}|0\rangle), z^{\frac{1}{2}})$, that $\mathrm{Aut}_2\,\mathcal{O}$ normalizes the $\mathcal{O}^*_{odd}$ action. We thus obtain an action of the semi-direct product $\mathrm{Aut}_2\,\mathcal{O} \ltimes \mathcal{O}^*_{odd}$ on M^σ.

5.2.2 Torsors for $\mathcal{O}^*_{odd}$–Odd Line Bundles

We saw in section 5.1 that a line bundle $\mathcal{L}$ naturally gives rise to an $\mathcal{O}^*$–torsor $\widehat{\mathcal{L}}_x$. In this section we construct analogous torsors for $\mathcal{O}^*_{odd}$ from odd line bundles on a curve. Let C be a Riemann surface with with an involution σ_C. We will say that a degree 0 line bundle is *odd* if

$$\sigma_C^*(\mathcal{L}) \cong \mathcal{L}^* \tag{5.1}$$

Let us fix an odd $\mathcal{L}$ and a σ_C–equivariant isomorphism 5.1.

Definition 5.2.2. Let $U \subset C$ be a σ_C–invariant open set. We say that $s \in \Gamma(U, \mathcal{L})$, $\mathcal{L}$ odd, is *special* if

$$s \otimes \sigma_C^*(s) = 1 \in \mathcal{O}_C$$

where we are using the isomorphism 5.1. We denote the space of special sections on U by $\Gamma_s(U, \mathcal{L}) \subset \Gamma(U, \mathcal{L})$.

Let $p \in C$ be a fixed point for σ_C, and pick a σ_C–special coordinate $z^{\frac{1}{2}}$ at p, so that $\sigma_C : z^{\frac{1}{2}} \mapsto -z^{\frac{1}{2}}$ on $\mathcal{D}_p^\times$. Let $h(z^{\frac{1}{2}}) \in \mathbb{C}[\mathcal{D}_p] \cong \mathbb{C}[[z^{\frac{1}{2}}]]$. If $s \in \Gamma_s(\mathcal{D}_p, \mathcal{L})$, then $hs \in \Gamma_s(\mathcal{D}_p, \mathcal{L})$ iff $h(z^{\frac{1}{2}})h(-z^{\frac{1}{2}}) = 1$. Definition 5.2.2 thus implies that $\Gamma_s(\mathcal{D}_p, \mathcal{L})$, the set of special trivializations of $\mathcal{L}|_{\mathcal{D}_p^\times}$, is an $\mathcal{O}_{odd}^*$–torsor.

Definition 5.2.3. Let $\widehat{\mathcal{L}}_{s,p}$ denote the set of pairs $(z^{\frac{1}{2}}, s)$, where $z^{\frac{1}{2}}$ is a special coordinate at p, and $s \in \Gamma_s(\mathcal{D}_p, \mathcal{L})$ is a special trivialization of $\mathcal{L}$ on $\mathcal{D}_p$.

$\widehat{\mathcal{L}}_{s,p}$ is a torsor for the semi-direct product $\operatorname{Aut}_2 \mathcal{O} \ltimes \mathcal{O}_{odd}^*$, and so we can form the twist

$$\mathcal{M}_p^{\sigma,\mathcal{L}} = \widehat{\mathcal{L}}_{s,p} \underset{\operatorname{Aut}_2 \mathcal{O} \ltimes \mathcal{O}_{odd}^*}{\times} M^\sigma$$

5.3 The Bundle $\mathcal{V}_X^{\mathcal{L},\sigma}$

Let C be as in the last section, and let $\mathcal{L}$ be an odd line bundle on C. In this section we extend the equivariant structure 3.8 on the bundle $\mathcal{V}_C$ to the bundle $\mathcal{V}_C^{\mathcal{L}}$, incorporating internal Heisenberg symmetries. Recall

$$\mathcal{V}_C^{\mathcal{L}} \stackrel{def}{=} \widehat{\mathcal{L}} \underset{\operatorname{Aut} \mathcal{O} \ltimes \mathcal{O}^*}{\times} V$$

so that the fiber of $\mathcal{V}^{\mathcal{L}}$ at $p \in C$ can be identified with the set of triples (A, z, s) where $A \in V, (z, s) \in \widehat{\mathcal{L}}_p$, modulo the action of $\operatorname{Aut} \mathcal{O} \ltimes \mathcal{O}^*$. We define an equivariant structure on $\mathcal{V}_C^{\mathcal{L}}$ as follows:

$$(p, (A, z, s)) \mapsto (\sigma_C(p), (\sigma_V(A), z \circ \sigma_C^{-1}, \sigma_C^*(s^{-1}))) \tag{5.2}$$

Here, $z \circ \sigma_C^{-1}$ is the coordinate induced at $\sigma_C(p)$ by composition with σ_C^{-1}, and $s^{-1} \in \Gamma(\mathcal{D}_p, \sigma_C^*(\mathcal{L}))$ satisfies $s \otimes s^{-1} = 1$, where we have used the isomorphism 5.1. This action lifts the action of σ_C on C.

Let $\overset{\circ}{C}, \overset{\circ}{X}$ be as in section 3.3.2. Using the equivariant structure 5.2, $\mathcal{V}^{\mathcal{L}}_{\overset{\circ}{C}}$ descends to a bundle $\mathcal{V}^{\mathcal{L},\sigma}_{\overset{\circ}{X}}$ on $\overset{\circ}{X}$. More explicitly,

$$\mathcal{V}^{\mathcal{L},\sigma}_{\overset{\circ}{X}} \overset{def}{=} \widehat{\mathcal{L}}_{\overset{\circ}{C}} \underset{(\mathrm{Aut}\,\mathcal{O}\times\mathbb{Z}/2\mathbb{Z})\ltimes\mathcal{O}^*}{\times} V$$

5.4 $(V, \mathcal{L})$–modules along σ_C–orbits

This section is the analogue of section 3.3.3 in the case of augmented Heisenberg symmetries. Here H has order 2, so H–orbits on C consist of either 1 or 2 points.

Let $x \in X$, a let $\{M_p^{h_p}\}_{p\in\nu^{-1}(x)}$ be a collection of V–modules satisfying properties 1 through 4 of section 3.3.3. The T–operators induce an equivalence relation on $\coprod_{p\in\nu^{-1}(x)} \mathcal{M}_p^{\mathcal{L},h_p}$. We will denote the set of equivalence classes by $\mathcal{M}_x^{\mathcal{L}}$.

Definition 5.4.1. We call $\mathcal{M}_x^{\mathcal{L}}$ a $(V, \mathcal{L})$*–module along* $\nu^{-1}(x)$.

5.5 TVOPs as Sections of $\mathcal{V}_X^{\mathcal{L},\sigma,*}$

We observe that a section of $\mathcal{V}^{\mathcal{L},\sigma,*}_{\overset{\circ}{X}}$ over $U \subset \overset{\circ}{X}$ is the same as a σ_C–equivariant section of $\mathcal{V}^{\mathcal{L},*}_{\overset{\circ}{C}}$ over $\nu^{-1}(U)$, and likewise for $\mathcal{V}^{H,*}_{\overset{\circ}{X}}$. Thus, to define a section of $\mathcal{V}^{\mathcal{L},\sigma,*}_{\overset{\circ}{X}}$ on $\mathcal{D}_x^\times$ is to define a σ_C–equivariant section of $\mathcal{V}^{\mathcal{L},*}_{\overset{\circ}{C}}$ on $\coprod_{p\in\nu^{-1}(x)} \mathcal{D}_p^\times$.

Remark 7. Even when $x \in X\backslash\overset{\circ}{X}$, the completion $\mathcal{V}^{\mathcal{L},\sigma}_{\overset{\circ}{X}}|_{\mathcal{D}_x^\times}$ exists, so we may speak of the restriction of $\mathcal{V}^{\mathcal{L},\sigma}_{\overset{\circ}{X}}$ to $\mathcal{D}_x^\times$ in the sense of Appendix A.

Let $p \in \nu^{-1}(x)$, and let $(z_p^{\frac{1}{N}}, s_p)$ be a pair, where $z_p^{\frac{1}{N}}$ is an h_p–special coordinate at p, and s_p is a special section $\mathcal{L}$ (if $\nu^{-1}(x)$ consists of two points, then this is just a pair (z_p, s_p) where z_p is (any) formal coordinate at p and s_p (any) element of $\Gamma(\mathcal{D}_p, \mathcal{L})$). Assume for a moment that $z_p^{\frac{1}{N}}$ and s_p extend non-vanishing to some neighborhood W of p. Then if $q \in W, q \neq p$, $z_p - z_p(q)$ is a coordinate at q, and s_p is a trivialization in a neighborhood of q. This implies that the map

$$\iota_{z_p,s_p} : V \times \mathcal{D}_p^\times \mapsto \mathcal{V}^{\mathcal{L}}_{\overset{\circ}{C}}|_{\mathcal{D}_p^\times}$$

given by

$$\iota_{z_p,s_p}(A,q) = (A, z_p - z_p(q), s_p(q))$$

is a trivialization of $\mathcal{V}^{\mathcal{L}}_{\overset{\circ}{C}}$ on $\mathcal{D}^{\times}_p$.

We have the following analogue of Theorem 5.1.1

Theorem 5.5.1. *Let $x \in X$, and $\mathcal{M}^{\mathcal{L}}_x$ a $(V,\mathcal{L})$–module along $\nu^{-1}(x)$. For each $p \in \nu^{-1}(x)$, choose an h_p–special coordinate $z_p^{\frac{1}{N}}$ at p, and a special section s_p. Define an $\mathrm{End}(\mathcal{M}^{\mathcal{L}}_x)$–valued section $\mathcal{Y}^{\mathcal{M}^{\mathcal{L}}_x}$ of $\mathcal{V}^{\mathcal{L},*}_{\overset{\circ}{C}}$ on $\coprod_{p\in\nu^{-1}(x)} \mathcal{D}^{\times}_p$ by the formula*

$$\langle (z_p^{\frac{1}{N}}, s_p, \phi), \mathcal{Y}^{\mathcal{M}^{\mathcal{L}}_x}(\iota_{z_p,s_p}(A)) \cdot (z_p^{\frac{1}{N}}, s_p, v)\rangle = \langle \phi, Y^{M^{h_p}}(A, z_p^{\frac{1}{N}}) \cdot v\rangle. \tag{5.3}$$

Then this section $\mathcal{Y}^{\mathcal{M}^{\mathcal{L}}_x}$ is independent of the choice of special coordinate $z_p^{\frac{1}{N}}$ and special section s_p on each $\mathcal{D}^{\times}_p$. Furthermore, it is σ_C–equivariant.

Proof. This is proved in the same way as Theorem 3.3.1. □

5.6 Conformal Blocks and Coinvariants

Theorem 5.5.1 allows us to define conformal blocks and coinvariants in the presence of augmented Heisenberg symmetries. We proceed in a manner analogous to that in section 4.1

Let $\{x_i\}_{i=1\cdots m}$ be a collection of distinct points of X, which contains all of the branch points of ν. Let $\{\mathcal{M}^{\mathcal{L}}_{x_i}\}_{i=1\cdots m}$ be a collection of $(V,\mathcal{L})$–modules, such that $\mathcal{M}^{\mathcal{L}}_{x_i}$ is a module along $\nu^{-1}(x_i)$. Let

$$\mathcal{F}^{\mathcal{L}} = \bigotimes_{i=1}^{m} \mathcal{M}^{\mathcal{L}}_{x_i}$$

Given $\phi \in \mathcal{F}^{\mathcal{L},*}$, $A_i \in \mathcal{M}^{\mathcal{L}}_{x_i}$,

$$\langle \phi, A_1 \otimes \cdots \otimes \mathcal{Y}^{\mathcal{M}^{\mathcal{L}}_{x_i}} \cdot A_i \otimes \cdots \otimes A_m\rangle \tag{5.4}$$

is a section of $\mathcal{V}^{\mathcal{L},\sigma,*}_{\overset{\circ}{X}}$ on $\mathcal{D}^{\times}_{x_i}$.

Definition 5.6.1. The space of conformal blocks

$$\mathcal{C}^{\mathcal{L}}_V(X, \{x_i\}, \mathcal{M}^{\mathcal{L}}_{x_i})_{i=1\cdots m}$$

is by definition the vector space of all linear functionals $\phi \in \mathcal{F}^{\mathcal{L},*}$ such that for any $A_i \in \mathcal{M}^{\mathcal{L}}_{x_i}$, the sections 5.4 can be extended to the *same* section of $\mathcal{V}^{\mathcal{L},\sigma,*}_{\overset{\circ}{X}}$, regular over $X\backslash\{x_i\}$

Equivalently, we can formulate this in terms of the cover C:

Definition 5.6.2. The space of conformal blocks

$$\mathcal{C}^{\mathcal{L}}_V(X, \{x_i\}, \mathcal{M}^{\mathcal{L}}_{x_i})_{i=1\cdots m}$$

is by definition the vector space of all linear functionals $\phi \in \mathcal{F}^{\mathcal{L},*}$ such that for any $A_i \in \mathcal{M}^{\mathcal{L}}_{x_i}$, the sections 5.4 can be extended to the *same* σ_C–equivariant section of $\mathcal{V}^{\mathcal{L},*}_C$, regular over $C\backslash\{\nu^{-1}(x_i)\}$

Remark. Observe that in this definition *all* branch points of ν are required to carry module insertions. □

Most of the results of Chapter 4 have obvious analogues in the case of augmented Heisenberg symmetries. In particular, in the case of $V = \pi$, we have a simpler description of the space of conformal blocks and coinvariants. Consider $\Gamma_s(C\backslash\{\nu^{-1}(x_i)\}, \mathcal{L})$, viewed as an Abelian Lie algebra. If $\nu^{-1}(x_i)$ consists of a single point p, $z^{\frac{1}{2}}$ is a special coordinate at p, $s_i \in \Gamma_s(\mathcal{D}_p, \mathcal{L})$, and $\psi \in \Gamma_s(C\backslash\{\nu^{-1}(x_i)\}, \mathcal{L})$, then ψ/s_i is an odd function on $\mathcal{D}^\times_p$, and expressed in $z^{\frac{1}{2}}$ will have only odd powers. Sending $z^k \to b_k \in \mathcal{H}^\sigma, k \in \frac{1}{2} + \mathbb{Z}$, we obtain a Lie algebra embedding $\Gamma_s(C\backslash\{\nu^{-1}(x_i)\}, \mathcal{L}) \to \mathcal{H}^\sigma$. If $\nu^{-1}(x)$ consists of two points p, p', we similarly obtain an embedding $\Gamma_s(C\backslash\{\nu^{-1}(x_i)\}, \mathcal{L}) \to \mathcal{H} \oplus \mathcal{H}$ by expanding ψ at each of p, p'. This way, $\Gamma_s(C\backslash\{\nu^{-1}(x_i)\}, \mathcal{L})$ acts on all $(V, \mathcal{L})$–modules along $\nu^{-1}(x_i)$. The following theorem is proved in the same way as Theorem 4.2.1.

Theorem 5.6.1.

$$\mathcal{C}^{\mathcal{L}}_V(X, \{x_i\}, \mathcal{M}^{\mathcal{L}}_{x_i})_{i=1\cdots m} \cong \mathrm{Hom}_{\Gamma_s(C\backslash\{\nu^{-1}(x_i)\},\mathcal{L})}(\mathcal{F}, \mathbb{C}) \tag{5.5}$$

and

$$\mathcal{H}^{\mathcal{L}}_V(X, \{x_i\}, \mathcal{M}^{\mathcal{L}}_{x_i})_{i=1\cdots m} \cong \mathcal{F}/\Gamma_s(C\backslash\{\nu^{-1}(x_i)\}, \mathcal{L}) \cdot \mathcal{F} \tag{5.6}$$

Chapter 6

Localization and Prym Varieties

One of the applications of Conformal Field Theory to algebraic geometry is the construction of sheaves on moduli spaces. This construction is realized in terms of a so called *localization functor*. Given a vertex algebra V, this functor sends the category of V–modules to the category of (twisted) D–modules on a certain moduli space $\mathfrak{M}_V$. When V is the Virasoro vertex algebra, $\mathfrak{M}_V$ is $\mathfrak{M}_g$, the moduli of curves of genus g. When $V = V_k(\mathfrak{g})$, the affine vacuum module attached to the affine algebra $\hat{\mathfrak{g}}$, $\mathfrak{M}_V$ is $Bun_G(X)$, the moduli stack of $G-bundles$ on a fixed Riemann surface X. In this context, the moduli space $\mathfrak{M}_V$ can be viewed as a kind of flag manifold for the vertex algebra V (or an underlying infinite-dimensional Lie algebra).

In this chapter, we construct a localization functor from the category of representations of $\mathcal{H}^\sigma$ to the category of twisted D–modules on a generalized Prym variety. The Prym variety is associated to to a curve C with an involution σ_C having fixed points.

Since $\mathcal{H}^\sigma$–modules are just σ–twisted π–modules, this leads us to conjecture the existence of a "twisted localization functor", which for a vertex algebra V with automorphism σ, sends σ–twisted V–modules to twisted D–modules on some "twisted" moduli space $\mathfrak{M}_V^\sigma$. In this section we treat the case when $V = \pi$, $\mathfrak{M}_V = Jac(C)$, and $\mathfrak{M}_V^\sigma = Prym(C)$.

6.1 Harish-Chandra Pairs and Actions

The construction of our twisted localization functor makes use of the formalism of Harish-Chandra localization (see [14] etc.). We briefly review this formalism following [14].

Definition 6.1.1. A *Harish-Chandra pair* is a pair $(\mathfrak{g}, K)$ where $\mathfrak{g}$ is a Lie algebra and K is a Lie group, equipped with the following data: an embedding $\mathfrak{k} \subset \mathfrak{g}$ of the Lie algebra $\mathfrak{k}$ of K into $\mathfrak{g}$, and an action Ad of K on $\mathfrak{g}$ compatible with the adjoint action of K on $\mathfrak{k} \subset \mathfrak{g}$ and the action of $\mathfrak{k}$ on $\mathfrak{g}$. A *Harish-Chandra*–module is a vector space V together with compatible actions of $\mathfrak{g}$ and K.

We also review what it means for a variety to carry an action of a Harish-Chandra pair:

Definition 6.1.2. Let Z be a complex variety. A $(\mathfrak{g}, K)$–action on Z is the data of an action of $\mathfrak{g}$ on Z (that is, a homomorphism α from $\mathfrak{g}$ to the tangent sheaf Θ_Z), together with an action of K on Z. The two actions are required to satisfy the following compatibility conditions.

- the differential of the K–action is the restriction of the $\mathfrak{g}$–action to $\mathfrak{k} = Lie(K)$;
- the action α intertwines the representation of K on $\mathfrak{g}$ with the action of K on $\Theta_Z : \alpha(Ad_k(a)) = k\alpha(a)k^{-1}$.

A $(\mathfrak{g}, K)$–action is called transitive if the map $\mathfrak{g} \otimes \mathcal{O}_Z \to \Theta_Z$ is surjective, and simply transitive if this map is an isomorphism.

6.2 Localization

In this section, we give a brief outline of the construction of the localization functor following [14]. Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair as above, where $\mathfrak{g}$ is a complex Lie algebra and K a complex algebraic group. Let S be a complex variety, and $p : Z \to S$ a principal K–bundle on S. Suppose that we are given a transitive $(\mathfrak{g}, K)$ action on Z

extending the fibrewise K–action on $Z \to S$. In other words, we are given a surjective Lie algebroid homomorphism

$$\alpha : \mathfrak{g} \otimes \mathcal{O}_Z \to \Theta_Z$$

Given a $(\mathfrak{g}, K)$–module V, we obtain an action of the Lie algebroid $\mathfrak{g} \otimes \mathcal{O}_Z$ on the sheaf $V \otimes \mathcal{O}_Z$. The sheaf

$$V_{stab} = V \otimes \mathcal{O}_Z / ker(\alpha) \cdot (V \otimes \mathcal{O}_Z)$$

carries an action of $\Theta_Z \cong \mathfrak{g} \otimes \mathcal{O}_Z / ker(\alpha)$, and is therefore a $\mathcal{D}_Z$–module. Moreover, V_{stab} is strongly K–equivariant, and so descends to a $\mathcal{D}_S$–module on S. This construction gives rise to a functor

$$\Delta : (\mathfrak{g}, K) - mod \to \mathcal{D}_S - mod$$

which we call the *localization functor.* Δ has as right-adjoint the functor $\hat{\Gamma}$ of global sections:

$$\hat{\Gamma} : \mathcal{D}_S - mod \to (\mathfrak{g}, K) - mod$$

sending $\mathcal{M}$ to $\Gamma(Z, p^* \mathcal{M})$.

6.3 Central Extensions

We will require a slight extension of the construction in section 6.2 that allows us to localize Harish-Chandra modules for central extensions of $\mathfrak{g}$. Suppose $p : Z \to S$ carries a transitive $(\mathfrak{g}, K)$-action as before, and let V be a Harish-Chandra module for the pair $(\hat{\mathfrak{g}}, K)$, where $\hat{\mathfrak{g}}$ is a central extension of $\mathfrak{g}$. The central extension gives rise to the following short exact sequence of Lie algebroids on Z

$$0 \to \mathcal{O}_Z \cdot \mathbf{1} \to \hat{\mathfrak{g}} \otimes \mathcal{O}_Z \to \mathfrak{g} \otimes \mathcal{O}_Z \to 0 \tag{6.1}$$

Let us assume that this exact sequence splits over the subalgebroid $ker(\alpha) \subset \mathfrak{g} \otimes \mathcal{O}_Z$. We then have an embedding of Lie algebroids $ker(\alpha) \hookrightarrow \hat{\mathfrak{g}} \otimes \mathcal{O}_Z$. The quotient $\mathcal{T}$ of $\hat{\mathfrak{g}} \otimes \mathcal{O}_Z$ by $ker(\alpha)$ then fits into the following extension:

$$0 \to \mathcal{O}_Z \to \mathcal{T} \to \Theta_Z \to 0 \tag{6.2}$$

and carries a Lie algebroid structure. Define the sheaf

$$\hat{V}_{stab} = V \otimes \mathcal{O}_Z / ker\alpha \cdot (V \otimes \mathcal{O}_Z)$$

$\hat{V}_{stab}$ carries an action of $\mathcal{T}$ which does not factor through Θ_Z unless the central element $\mathbf{1}$ acts by 0. Suppose that $\mathbf{1}$ acts on V as the identity (if it acts by a different non-zero scalar, re-scale the extension to make it act by the identity). Then the quotient $\mathcal{D}'_Z$ of the enveloping algebroid $U(\mathcal{T})$ by the relation identifying $1 \in \mathcal{O}_Z \subset \mathcal{T}$ with the identity element of $U(\mathcal{T})$ acts on $\hat{V}_{stab}$. $\mathcal{D}'_Z$ is a sheaf of *twisted differential operators* on Z, or TDO for short. As in section 6.2, the construction of $\hat{V}_{stab}$ is K–equivariant, and so it descends to a $\mathcal{D}'_S$–module on S, also denoted $\Delta(V)$, where $\mathcal{D}'_S$ is the TDO on S corresponding to $\mathcal{D}'_Z$. We thus obtain a (twisted) localization functor

$$\Delta : (\hat{\mathfrak{g}}, K) - mod \to \mathcal{D}'_S - mod$$

6.4 Prym Varieties Associated to Involutions

We begin by introducing Prym varieties associated to involutions. Let C be smooth compact Riemann surface of genus $g > 0$ with an involution σ_C, $X = C/\langle\sigma_C\rangle$, and let $\pi : C \to X$ be the quotient map. π induces a map (pullback)

$$\pi^* : Jac(X) \to Jac(C)$$

(When $X = \mathbb{CP}^1$, $Jac(X)$ is the group with one element). The image of $Jac(X)$ under π^* forms an abelian subvariety of $Jac(C)$, and we define the associated Prym variety $Prym(C, \sigma_C)$ to be the complementary abelian variety to $\pi^*(Jac(X))$ in $Jac(C)$. An alternative, and for our purposes more useful definition of $Prym(C, \sigma_C)$, is as those line bundles $\mathcal{L} \in Jac(C)$ such that

$$\sigma_C^*(\mathcal{L}) \cong \mathcal{L}^*. \tag{6.3}$$

Our definition of Prym variety differs from the usual one in that we do not insist that the polarization induced on $Prym(C, \sigma_C)$ as a subvariety of $Jac(C)$ be a multiple of a principal polarization. In fact, for most of the following discussion, polarization issues will be unimportant. The important property is the behavior 6.3 of the line bundle with respect to σ_C. For example, according to our definition, if C is hyperelliptic and σ_C is the hyperelliptic involution, $Prym(C, \sigma_C) = Jac(C)$.

Let $L \to C \times Jac(C)$ denote the Picard line bundle. Recall that it has the property that $L|_{C\times\{\mathcal{L}\}} \cong \mathcal{L}$. The automorphism σ_C acts on $C \times Jac(C)$ by

$$\tilde{\sigma}_C : (p, \mathcal{L}) \to (\sigma_C(p), \mathcal{L})$$

Let $\bar{L} = L|_{C\times Prym(C,\sigma_C)}$. It then follows from 6.3 that

$$\tilde{\sigma}_C^*(\bar{L}) \otimes \bar{L} \cong \mathcal{O}_{C\times Prym(C,\sigma_C)} \tag{6.4}$$

We will fix such an isomorphism once and for all, and it will always be implicit in the following discussion. 6.4 induces isomorphisms 6.3 $\forall \mathcal{L} \in Prym(C, \sigma_C)$.

6.5 Prym Uniformization

6.5.1 The Harish-Chandra Pair $(\mathcal{H}^\sigma, \mathcal{O}^*_{odd})$

In this section we introduce the Harish-Chandra pair which is responsible for localization of twisted Heisenberg representations on the Prym variety.

Definition 6.5.1. Let $\mathcal{O}^*_{odd}$ (resp. $\mathcal{K}^*_{odd}$) denote the group-scheme of infinite type (resp. Ind group-scheme) whose R–points, for a commutative $\mathbb{C}$–algebra R, are those elements $\gamma(t^{\frac{1}{2}}) \in R[[t^{\frac{1}{2}}]]$ (resp. $R((t^{\frac{1}{2}}))$) such that $\gamma(t^{\frac{1}{2}}) \cdot \gamma(-t^{\frac{1}{2}}) = 1$.

Remark 8. The $\mathbb{C}$–points of $\mathcal{K}^*_{odd}$ are coincide with those of $\mathcal{O}^*_{odd}$.

We thus have $\mathcal{O}^*_{odd} \subset \mathcal{K}^*_{odd}$. Over $\mathbb{C}$,

$$Lie(\mathcal{K}^*_{odd}) = \overline{\mathcal{H}}^\sigma = \mathcal{H}^\sigma/\mathbb{C}\cdot K$$
$$Lie(\mathcal{O}^*_{odd}) = \mathcal{H}^\sigma_+$$

Since $\mathcal{O}^*_{odd}$ is nilpotent, all highest-weight representations of $\mathcal{H}^\sigma$ are automatically integrable to $\mathcal{O}^*_{odd}$ representations.

6.5.2 $\widehat{Prym}(C, \sigma_C)$

Let C, σ_C be as in section 6.4. We henceforth assume that σ_C has at least one fixed point, and denote the fixed points by $\{p_i\}_{i=1\cdots n}$. Let $\widehat{Prym}(C, \sigma_C)$ denote the moduli space of tuples

$$(\mathcal{L}, \{s_i\}_{i=1\cdots n})$$

where $\mathcal{L} \in Prym(C, \sigma_C)$, and $s_i \in \Gamma_s(\mathcal{L}, \mathcal{D}_{p_i})$ is a special trivialization of $\mathcal{L}|\mathcal{D}_{p_i}$. The map

$$\begin{aligned} \widehat{Prym}(C, \sigma_C) &\to Prym(C, \sigma_C) \\ (\mathcal{L}, \{s_i\}_{i=1\cdots n}) &\to \mathcal{L} \end{aligned}$$

is a principal bundle for the group $\prod_{i=1}^n \mathcal{O}^*_{odd}$, since two special trivializations at a σ_C–fixed point p_i differ by an element of $\mathcal{O}^*_{odd}$. We now have the following analogue of Theorem 17.1.2 in [14]:

Theorem 6.5.1. *$\widehat{Prym}(C, \sigma_C)$ carries a transitive action of the Harish-Chandra pair*

$$(\oplus_{i=1}^n \overline{\mathcal{H}}_i^\sigma, \prod_{i=1}^n \mathcal{O}^*_{odd,i})$$

compatible with the action of $\prod_{i=1}^n \mathcal{O}^*_{odd,i}$ along the fibers of the map $\widehat{Prym}(C, \sigma_C) \to Prym(C, \sigma_C)$

Proof. We construct an action of the group $\prod_{i=1}^n {}_{odd,i}$ on $\widehat{Prym}(C, \sigma_C)$, and hence the Lie algebra $\oplus_{i=1}^n \overline{\mathcal{H}}_i^\sigma$, as follows. Let $(\mathcal{L}, \{s_i\}_{i=1\cdots n})$ be an R–point of $\widehat{Prym}(C, \sigma_C)$, and $g = (g_i)_{i=1\cdots n}$ an R–point of $\prod_{i=1}^n {}_{odd,i}$. Define a new line bundle $\mathcal{L}_g$ as follows. The set $\Gamma_s(U, \mathcal{L}_g)$ of special sections over a σ_C–invariant open subset U not containing any of the p_i is by definition equal to the set $\Gamma_s(U, \mathcal{L})$. On the other hand, if $p_i \in U$ for some i, then $\Gamma_s(U, \mathcal{L}_g)$ is by definition the special sections ψ of $\mathcal{L}$ on $U \backslash \{p_i\}$ satisfying the condition that $g_i \times \frac{\psi_{p_i}}{s_i} \in \mathcal{O}^*_{odd,i}$. We see that $g : (\mathcal{L}, \{s_i\}) \to (\mathcal{L}_g, \{s_i\})$ defines an action of $\prod_{i=1}^n {}_{odd,i}$ on $\widehat{Prym}(C, \sigma_C)$, extending the fibrewise action of $\prod_{i=1}^n \mathcal{O}^*_{odd,i}$ by changes of trivialization. The proof of transitivity of this action is the same as that in Theorem 17.1.2 of [14]. $\square$

Remark 9. The arguments of this section go through if we choose only one σ_C–fixed point p, and consider the $\mathcal{O}^*_{odd}$–principal bundle $\widehat{Prym_p(C, \sigma_C)} \to \widehat{Prym(C, \sigma_C)}$ whose fiber over $\mathcal{L} \in \widehat{Prym(C, \sigma_C)}$ consists of special trivializations near p only.

6.6 Prym Localization

Theorem 6.5.1 implies that we obtain a localization functor

$$\Delta : (\oplus_{i=1}^n \mathcal{H}_i^\sigma, \prod_{i=1}^n \mathcal{O}^*_{odd,i}) - \text{Mod} \to \mathcal{D}'_{Prym,\{p_i\}} - \text{Mod}$$

as well as (following Remark 9), localization functors

$$\Delta_p : (\mathcal{H}^\sigma, \mathcal{O}^*_{odd}) - \text{Mod} \to \mathcal{D}'_{Prym,p} - \text{Mod}$$

for each fixed point p of σ_C.

Remark 10. The identity of the TDO's $\mathcal{D}'_{Prym,\{p_i\}}$ and $\mathcal{D}'_{Prym,p}$ in general is unknown at this time. I conjecture that these are sheaves of differential operators acting on powers of the Θ–line bundle on $Prym(C, \sigma_C)$.

Finally, we recall the description of the fibers of the localization $\Delta(V)$ of a $(\hat{\mathfrak{g}}, K)$–module V (viewed as an $\mathcal{O}_S$–module) given in 16.2.14 of [14]. We use the notation of section 6.2. Let $s \in S$, and Z_s the fiber of the K–torsor Z over s. Let $\mathfrak{g}_s = Z_s \underset{K}{\times} \mathfrak{g}$. $\mathfrak{g}_s$ acts on $\mathcal{V}_s = Z_s \underset{K}{\times} V$. The stabilizers of the points of $Z_s \subset Z$ in $\mathfrak{g}$ give rise to a subalgebra $\mathfrak{g}^s_{stab}$ of $\mathfrak{g}_s$. Lemma 16.2.9 of [14] states that the fiber of $\Delta(V)$ at $s \in S$ is canonically isomorphic to the space of coinvariants $\mathcal{V}_s / \mathfrak{g}^s_{stab} \cdot \mathcal{V}_s$.

In this case, for $s = \mathcal{L} \in Prym(C, \sigma_C)$, $\mathfrak{g}^s_{stab}$ is the Abelian Lie algebra $\Gamma_s(\overset{\circ}{C}, \mathcal{L})$, where $\overset{\circ}{C} = C \backslash \{p_i\}_{i=1\cdots n}$. Thus, given a module V for the Harish-Chandra pair

$$(\oplus_{i=1}^n \mathcal{H}_i^\sigma, \prod_{i=1}^n \mathcal{O}^*_{odd,i})$$

the fiber of $\Delta(V)$ at $\mathcal{L} \in Prym(C, \sigma_C)$ is isomorphic to $\mathcal{V}^{\mathcal{L}} / \Gamma_s(\overset{\circ}{C}, \mathcal{L}) \cdot \mathcal{V}^{\mathcal{L}}$, which in turn is $\mathcal{H}_V^{\mathcal{L}}(X, \{x_i\}, \mathcal{M}_{x_i}^{\mathcal{L}})_{i=1\cdots m}$, the space of coinvariants defined in 5.6.

Part II

Second Part

Chapter 7

Wakimoto Modules

7.1 Introduction

Wakimoto modules are an important family of highest weight representations of affine Lie algebras. They were first constructed in [25] for $\hat{\mathfrak{sl}}_2$, and in [8] for an arbitrary untwisted affine lie algebra $\hat{\mathfrak{g}}$. The Wakimoto module $W_{\chi,k}$ of level k and highest weight χ has the same character as the corresponding Verma module $M_{k,\chi}$, yet may possess a different composition series. Wakimoto modules have important applications in conformal field theory, representation theory, integrable systems, as well as other areas.

In this part of the thesis, we construct Wakimoto modules for twisted affine Lie algebras. These should have analogous applications, for example in conformal field theory, where they can be used to bosonize orbifold models. We compute explicit realizations of these representations for the twisted affine lie algebra $A_2^{(2)}$.

It is our goal to cast the Wakimoto construction in the language of vertex algebras and their modules. In the untwisted case, the picture is as follows. To $\hat{\mathfrak{g}}$ at level k, we can associate the vacuum module vertex algebra $V_k(\mathfrak{g})$. The Wakimoto construction maps $V_k(\mathfrak{g})$ to a tensor product of two vertex algebras $M \otimes \pi^{k+h^\vee}$, where M is a $\beta\gamma$ system, and $\pi^{k+h^\vee}$ a Heisenberg vertex algebra. $\pi^{k+h^\vee}$ has a family of modules $\pi_\chi^{k+h^\vee}$, and so $W_{\chi,k} = M \otimes \pi_\chi^{k+h^\vee}$, an $M \otimes \pi^{k+h^\vee}$ module, inherits a structure of $V_k(\mathfrak{g})$ module via the homomorphism $\iota : V_k(\mathfrak{g}) \to M \otimes \pi^{k+h^\vee}$.

In the twisted case, a similar picture holds. Let $\sigma_{\mathfrak{g}}$ be a diagram automorphism of $\mathfrak{g}$, and let $\hat{\mathfrak{g}}^{\sigma}$ be the corresponding twisted affine Lie algebra. M has a twisted module M^{σ}, and $\pi^{k+h^\vee}$ has a family of twisted modules $\pi_{\chi}^{k+h^\vee,\sigma}$. Thus we obtain a family of twisted $M \otimes \pi^{k+h^\vee}$ - modules $W_{k,\chi}^{\sigma} = M^{\sigma} \otimes \pi_{\chi}^{k+h^\vee,\sigma}$. $W_{k,\chi}^{\sigma}$, when viewed as a twisted $V_k(\mathfrak{g})$ - module via the embedding $\iota : V_k(\mathfrak{g}) \to M \otimes \pi^{k+h^\vee}$, has the the structure of a $\hat{\mathfrak{g}}^{\sigma}$ module of level k.

Once Wakimoto modules for twisted affine algebras have been constructed, they can be used to prove the Kac-Kazhdan conjecture about characters of irreducible modules at the critical level $k = -h^\vee$. In the final sections of this paper we state and prove this conjecture in the twisted case. Our proof follows the one in [13] for untwisted affine algebras.

I would like to thank Edward Frenkel for suggesting this problem and for many very valuable discussions.

7.2 Twisted Affine Lie Algebras

In this section we briefly review the definition of twisted affine algebras following [21]. We begin by introducing some standard notation. Given a Kac-Moody algebra $\mathfrak{k}$, we use the notation $\Lambda(\mathfrak{k})$, $\Delta(\mathfrak{k})$, $\Delta^+(\mathfrak{k})$, $\Delta^+_{re}(\mathfrak{k})$, and $\Delta^+_{im}(\mathfrak{k})$, to denote respectively, the root lattice of $\mathfrak{k}$, the set of non-zero roots of $\mathfrak{k}$, the positive roots, the positive real roots, and the positive imaginary roots.

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra, possessing a non-trivial diagram automorphism $\sigma_{\mathfrak{g}}$ of finite order N. $N = 2$ when $\mathfrak{g} = A_n, D_n, E_6$ and $N = 3$ when $\mathfrak{g} = D_4$. We will use the notation $\sigma_{\mathfrak{g}}$ both to refer to the action of the automorphism on $\mathfrak{g}$, as well as the action on the root lattice of $\mathfrak{g}$. Let $\epsilon = e^{\frac{2\pi i}{N}}$, and let us write

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}/N\mathbb{Z}} \mathfrak{g}_j$$

where $\mathfrak{g}_j$ is the ϵ^j eigenspace of σ. Furthermore, if $\mathfrak{l} \subset \mathfrak{g}$ is a subspace in $\mathfrak{g}$, let

$\mathfrak{l}_i = \mathfrak{l} \cap \mathfrak{g}_i$. Let

$$L\mathfrak{g}^\sigma = \mathbb{C} \cdot d \oplus \bigoplus_{j=0}^{N-1} \mathfrak{g}_j \otimes t^{\frac{j}{N}} \mathbb{C}((t))$$

with Lie bracket

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m}$$

$$[d, x \otimes t^n] = n(x \otimes t^n)$$

Let $\check{r}$ denote the dual Coxeter number of $\mathfrak{g}$. $L\mathfrak{g}^\sigma$ has a central extension $\hat{\mathfrak{g}}^\sigma = L\mathfrak{g}^\sigma \oplus \mathbb{C} \cdot K$ defined by the cocycle

$$\omega(x \otimes t^n, y \otimes t^m) = n(x, y)\delta_{n,-m} K$$

where $(x, y) = \frac{1}{2\check{r}} Tr(ad(x)ad(y))$ denotes the normalized invariant inner product on $\mathfrak{g}$. In our discussion we will also use the untwisted loop algebra

$$L\mathfrak{g} = \mathbb{C} \cdot d \oplus \mathfrak{g} \otimes \mathbb{C}((t))$$

with the same commutation relations as above, and its central extension $\hat{\mathfrak{g}}$ with respect to the cocycle ω. Given $X \in \mathfrak{g}$, we will henceforth use the notation X_n to denote $X \otimes t^n$. The automorphism $\sigma_{\mathfrak{g}}$ of $\mathfrak{g}$ induces an automorphism $\sigma_{L\mathfrak{g}}$ of $L\mathfrak{g}$ and (which extends to $\hat{\mathfrak{g}}$), which acts by $\sigma_{L\mathfrak{g}}(X_n) = \sigma_{\mathfrak{g}}(X)_n$. We denote the dual Coxeter number of $\hat{\mathfrak{g}}^\sigma$ by $h^\vee$. It is equal to the dual Coexter number of $\hat{\mathfrak{g}}$.

Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the triangular decomposition of $\mathfrak{g}$. $\sigma_{\mathfrak{g}}$ preserves it, and we have $L\mathfrak{g}^\sigma = L\mathfrak{n}_-^\sigma \oplus L\mathfrak{h}^\sigma \oplus L\mathfrak{n}_+^\sigma$. The Lie subalgebra $\mathfrak{g}_0$ fixed by $\sigma_{\mathfrak{g}}$ is simple, and each $\mathfrak{g}_j$, $(j = 0, \cdots, N-1)$ is an irreducible representation of $\mathfrak{g}_0$. In the case $N = 3$, the $\mathfrak{g}_0$ representations $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are equivalent. Let Δ_j denote the set of non-zero weights occurring in $\mathfrak{g}_j$ (viewed as a $\mathfrak{g}_0$ representation), and $\mathfrak{g}_{j,\alpha}$, $\alpha \in \Delta_j$, the corresponding weight space. We have that $\Delta_j \subset \Lambda(\mathfrak{g}_0)$, $dim(\mathfrak{g}_{j,\alpha}) = 1$, and $[\mathfrak{g}_{i,\alpha}, \mathfrak{g}_{j,\beta}] \subset g_{(i+j)\, mod\, N, \alpha+\beta}$. Let us fix a basis $E_{j,\alpha}$ for $\mathfrak{g}_{j,\alpha}, \alpha \in \Delta_j$, and $H_{j,a}$ for $\mathfrak{h}_j$, $a = 1, \cdots, dim(\mathfrak{h}_j)$. Then $\{E_{n,(j,\alpha)}, H_{n,(j,a)}, d\}$, $n \in \mathbb{Z}$ (resp. $n \in \frac{j}{N} + \mathbb{Z}$) forms a basis for $L\mathfrak{g}$ (resp. $L\mathfrak{g}^\sigma$).

Let $\mathfrak{n}_{j,\alpha} = \mathfrak{g}_{j,\alpha} \cap \mathfrak{n}_+$, and let $\Delta_j^+ = \{\alpha \in \Delta_j | \mathfrak{n}_{j,\alpha} \neq 0\}$ We therefore obtain that

$$L\mathfrak{n}_+^\sigma = \bigoplus_{j=0}^{N-1} \bigoplus_{\alpha \in \Delta_j^+} \mathfrak{n}_{j,\alpha} \otimes t^{\frac{j}{N}} \mathbb{C}((t))$$

Thus $E_{n,(j,\alpha)}$, $n \in \frac{j}{N} + \mathbb{Z}, \alpha \in \Delta_j^+$ forms a basis for $L\mathfrak{n}_+^\sigma$.

7.3 Vertex Algebras and Twisted Modules

For an introduction to vertex algebras see ([14], [19]). For a definition of twisted module see [6] and [1]. We will denote the vacuum vector by $|0\rangle$, and use $\sim$ to denote the singular terms of the operator product expansion (OPE). If V is a vertex algebra, and $A \in V$ is a vector of conformal weight p, then we write $Y(A,z) = \sum_{n\in\mathbb{Z}} A[n]z^{-n-p}$ - i.e. $A[n]$ denotes the Fourier coefficient in front of z^{-n-p}. It is was shown in ([17], [10])that

$$V_k(\mathfrak{g}) = Ind_{\mathfrak{g}[[t]]}^{\hat{\mathfrak{g}}}\mathbb{C}_k$$

(where $\mathbb{C}_k$ denotes the one-dimensional representation on which $\mathfrak{g}[[t]]$ acts trivially and K acts by $k \in \mathbb{C}$) has a vertex algebra structure. Given $J \in \mathfrak{g}$, this structure assigns the field

$$J(z) = Y_{V_k(\mathfrak{g})}(J_{-1}|0\rangle, z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}$$

The OPE is as follows:

$$J^1(z)J^2(w) \sim \frac{[J^1,J^2](w)}{z-w} + \frac{k(J^1,J^2)}{(z-w)^2} \tag{7.1}$$

The automorphism $\sigma_{\mathfrak{g}}$ of $\mathfrak{g}$ induces a vertex algebra automorphism $\sigma_{V_k(\mathfrak{g})}$ of $V_k(\mathfrak{g})$ which acts by

$$\sigma_{V_k(\mathfrak{g})}(J^1_{n_1}\cdots J^k_{n_k}|0\rangle) \to \sigma_{\mathfrak{g}}(J^1)_{n_1}\cdots\sigma_{\mathfrak{g}}(J^k)_{n_k}|0\rangle, \qquad n_i < 0$$

To $J \in \mathfrak{g}_j$ we can also assign the twisted field

$$J^\sigma(z) = Y^\sigma_{V_k(\mathfrak{g})}(J_{-1}|0\rangle, z) = \sum_{n\in\frac{j}{N}+\mathbb{Z}} J_n z^{-n-1}$$

If $J^1 \in \mathfrak{g}_i, J^2 \in \mathfrak{g}$, the OPE of the twisted fields is as follows:

$$J^{1,\sigma}(z)J^{2,\sigma}(w) \sim \frac{z^{-i/N}w^{i/N}[J^1,J^2]^\sigma(w)}{(z-w)} + k(J^1,J^2)\partial_w\frac{z^{-i/N}w^{i/N}}{(z-w)^2} \tag{7.2}$$

The commutation relations between the Fourier coefficients of fields can be recovered from the singular terms in the OPE.

7.4 Twisted Fock Spaces

We proceed to define various Bosonic vertex algebras and their twisted modules that will be used in the twisted Wakimoto realization.

7.4.1 The $\beta\gamma$ System

Let $\mathcal{A}$ (respectively $\mathcal{A}^\sigma$) denote the Heisenberg Lie algebra with generators $\{\tilde{a}^*_{n,(j,\alpha)}, \tilde{a}_{n,(j,\alpha)}, \mathbf{1}\}$ $, n \in \mathbb{Z}, j = 0, \cdots, N-1,\ \alpha \in \Delta_j^+$, (respectively $\{a^*_{n,(j,\alpha)},\ a_{m,(j,\alpha)}, \mathbf{1}\}$ $, n \in -\frac{j}{N} + \mathbb{Z}, m \in \frac{j}{N} + \mathbb{Z}, j = 0, \cdots, N-1, \alpha \in \Delta_j^+$) and commutation relations

$$[\tilde{a}_{n,(j,\alpha)}, \tilde{a}^*_{m,(k,\beta)}] = \delta_{n,-m}\delta_{j,k}\delta_{\alpha,\beta}\mathbf{1}$$

(respectively, the same commutation relations with $\tilde{a}, \tilde{a}^*$'s replaced with a, a^* 's). The element $\mathbf{1}$ is central. $\mathcal{A}$ (resp. $\mathcal{A}^\sigma$) has a "positive" abelian Lie subalgebra $\mathcal{A}_+$ (resp. $\mathcal{A}^\sigma_+$) spanned by $\tilde{a}^*_{n,(j,\alpha)}, n > 0,\ \tilde{a}_{m,(j,\alpha)}, m \geq 0$ (resp. $a^*_{n,(j,\alpha)}, n > 0,\ a_{m,(j,\alpha)}, m \geq 0$)

Let $\mathbf{C}$ denote the 1-dimensional representation of $\mathcal{A}_+ \oplus \mathbb{C}\cdot\mathbf{1}$ (resp. $\mathcal{A}^\sigma_+ \oplus \mathbb{C}\cdot\mathbf{1}$) on which $\mathcal{A}_+$ (resp. $\mathcal{A}^\sigma_+$) acts trivially, and $\mathbf{1}$ acts by 1, and let

$$M = Ind^{\mathcal{A}}_{\mathcal{A}_+\oplus\mathbb{C}\cdot\mathbf{1}}\mathbf{C}$$

and

$$M^\sigma = Ind^{\mathcal{A}^\sigma}_{\mathcal{A}^\sigma_+\oplus\mathbb{C}\cdot\mathbf{1}}\mathbf{C}$$

It is well-known that M has a vertex algebra structure (see [14]), generated in the sense of the reconstruction theorem (for the reconstruction theorem for vertex algebras see [19], [15]) by the field assignments (here $|0\rangle$ denotes the vacuum vector):

$$\tilde{a}^*_{(j,\alpha)}(z) = Y_M(\tilde{a}^*_{0,(j,\alpha)}|0\rangle, z) = \sum_{n\in\mathbb{Z}} \tilde{a}^*_{n,(j,\alpha)} z^{-n}$$

$$\tilde{a}_{(j,\alpha)}(z) = Y_M(\tilde{a}_{-1,(j,\alpha)}|0\rangle, z) = \sum_{n\in\mathbb{Z}} \tilde{a}_{n,(j,\alpha)} z^{-n-1}$$

M has a vertex algebra automorphism of finite order N, which we will denote σ_M, induced from the automorphism of $\mathcal{A}$ taking

$$\tilde{a}^*_{n,(j,\alpha)} \to \epsilon^{-j}\tilde{a}^*_{n,(j,\alpha)}$$

$$\tilde{a}_{n,(j,\alpha)} \to \epsilon^{j}\tilde{a}_{n,(j,\alpha)}$$

M^σ carries the structure of a σ_M - twisted M-module. The vertex operation

$$Y_M^\sigma : M \to End(M^\sigma)[[z^{\frac{1}{N}}, z^{-\frac{1}{N}}]]$$

is generated by the fields

$$a^*_{(j,\alpha)}(z) = Y_M^\sigma(\tilde{a}^*_{0,(j,\alpha)}|0\rangle, z) = \sum_{n\in\frac{-j}{N}+\mathbb{Z}} a^*_{n,(j,\alpha)} z^{-n}$$

$$a_{(j,\alpha)}(z) = Y_M^\sigma(\tilde{a}_{-1,(j,\alpha)}|0\rangle, z) = \sum_{n\in\frac{j}{N}+\mathbb{Z}} a_{n,(j,\alpha)} z^{-n-1}$$

7.4.2 The Free Boson

Let $\mathcal{H}$ (resp. $\mathcal{H}^\sigma$) denote the Heisenberg Lie algebra with generators $\{\tilde{b}_{n,(i,a)}, \mathbf{1}\}$, $n \in \mathbb{Z}$, $i = 0, \cdots, N-1$, $a = 1, \cdots, dim(\mathfrak{h}_i)$, (resp. $\{b_{n,(i,a)}, \mathbf{1}\}$, $n \in \frac{i}{N} + \mathbb{Z}$, $i = 0 \cdots N-1$, $a = 1, \cdots, dim(\mathfrak{h}_i)$) and commutation relations

$$[\tilde{b}_{n,(i,a)}, \tilde{b}_{m,(j,b)}] = n(H_{i,a}, H_{j,b})\delta_{n,-m}\mathbf{1}$$

(resp. $\tilde{b}$'s replaced by b's). The element $\mathbf{1}$ is central. $\mathcal{H}$ (resp. $\mathcal{H}^\sigma$) has an abelian subalgebra $\mathcal{H}_+$ (resp. $\mathcal{H}_+^\sigma$) spanned by $\tilde{b}_{n,(i,a)}$, $n \geq 0$ (resp. $b_{n,(i,a)}$, $n \geq 0$). Let $\mathbf{C}_r$ denote the one-dimensional representation of $\mathcal{H}_+ \oplus \mathbb{C} \cdot \mathbf{1}$ on which $\mathcal{H}_+$ acts trivially and $\mathbf{1}$ acts by r, and let

$$\pi^r = Ind^{\mathcal{H}}_{\mathcal{H}_+ \oplus \mathbb{C}\cdot\mathbf{1}} \mathbf{C}_r$$

Then it is well-known that π^r has a vertex algebra structure (see [14]), generated by the fields

$$\tilde{b}_{i,a}(z) = Y_\pi(\tilde{b}_{-1,(i,a)}|0\rangle, z) = \sum_{n\in\mathbb{Z}} \tilde{b}_{n,(i,a)} z^{-n-1}$$

Let $\chi \in \mathfrak{h}_0^*$, and let $c_a = \chi(H_{0,a})$. Let $\mathbf{C}_{r,\chi}$ denote the one-dimensional representation of $\mathcal{H}_+^\sigma$ where $b_{0,(0,a)}$ acts by c_a, all other generators of $\mathcal{H}_+^\sigma$ act by 0, and $\mathbf{1}$ acts by r. Let

$$\pi_\chi^{r,\sigma} = Ind^{\mathcal{H}^\sigma}_{\mathcal{H}_+^\sigma \oplus \mathbb{C}\cdot\mathbf{1}} \mathbf{C}_{r,\chi}$$

π^r has a vertex algebra automorphism of order N, which we will denote σ_π, that is induced from the automorphism of $\mathcal{H}$ sending $\tilde{b}_{n,(i,a)} \to \epsilon^i \tilde{b}_{n,(i,a)}$. Then $\pi^{r,\sigma}_\chi$ has a structure of σ_π - twisted π^r - module, where the vertex operation $Y^\sigma_\pi : \pi^r \to End(\pi^{r,\sigma}_\chi)[[z^{\frac{1}{N}}, z^{\frac{-1}{N}}]]$ is generated by the fields

$$b_{i,a}(z) = Y^\sigma_\pi(\tilde{b}_{-1,(i,a)}|0\rangle, z) = \sum_{n \in \frac{i}{N}+\mathbb{Z}} b_{n,(i,a)} z^{-n-1}$$

7.4.3 Tensor Products of Fock Spaces

$M \otimes \pi^r$ has a vertex algebra structure with the vertex operation

$$Y_{M\otimes\pi}(A \otimes B, z) = Y^(A, z) \otimes Y_\pi(B, z), \qquad A \in M, B \in \pi^r$$

and $\sigma_{M\otimes\pi} = \sigma_M \otimes \sigma_\pi$ is an automorphism of $M \otimes \pi^r$ of order N. $M^\sigma \otimes \pi^{r,\sigma}_\chi$ has the structure of a $\sigma_{M\otimes\pi}$ - twisted $M \otimes \pi$ - module, with the vertex operation given by

$$Y^\sigma_{M\otimes\pi}(A \otimes B, z) = Y^\sigma_M(A, z) \otimes Y^\sigma_\pi(B, z), \qquad A \in M, B \in \pi^r$$

7.5 The Twisted Wakimoto Construction

In this section, we prove the existence of twisted Wakimoto modules using the untwisted construction. By an embedding of vertex algebras we mean a vertex algebra homomorphism which is both injective and surjective. The following theorem follows from ([9], [12]) (see also [14] for the case $\mathfrak{g} = \mathfrak{sl}_2$)

Theorem 7.5.1. *(Untwisted Wakimoto Construction) (i) There exists an embedding of vertex algebras*

$$V_k(\mathfrak{g}) \xrightarrow{\iota} M \otimes \pi^{k+h^\vee} \tag{7.3}$$

for any $k \in \mathbb{C}$
(ii) When $k = -h^\vee$, there exists a vertex algebra homomorphism

$$V_{-h^\vee}(\mathfrak{g}) \xrightarrow{\kappa} M \tag{7.4}$$

This homomorphism is neither surjective nor injective.

Note that when $k = -h^\vee$ (critical level), $V_{-h^\vee}(\mathfrak{g})$ is not a conformal vertex algebra.

Remark: Given vector spaces V, W, each equipped with an action of $\mathbb{Z}/N\mathbb{Z}$, we will refer to a linear map $f : V \to W$ simply as "equivariant" if f commutes with this action.

$\mathbb{Z}/N\mathbb{Z}$ acts on $V_k(\mathfrak{g})$ via $\sigma_{V_k(\mathfrak{g})}$ and on $M \otimes \pi^{k+h^\vee}$ via $\sigma_{M\otimes\pi}$. We can therefore ask whether the vertex algebra homomorphisms ι and κ are equivariant. This is indeed the case:

Theorem 7.5.2. *ι and κ are equivariant vertex algebra homomorphisms.*

Proof. Let G be the simply-connected algebraic group with Lie algebra $\mathfrak{g}$. $\sigma_{\mathfrak{g}}$ descends to G, preserving B_-, the Borel subgroup with Lie algebra $\mathfrak{b}_-$. Thus $\sigma_{\mathfrak{g}}$ descends to the Flag Manifold G/B_-, and in particular preserves the big cell $N_+ \cdot B_- \cong \mathfrak{n}_+$. This induces an action of $\mathbb{Z}/N\mathbb{Z}$ on $Vect(N_+ \cdot B_-)$ - the Lie algebra of vector fields on $N_+ \cdot B_-$, denoted σ_{N_+}. It follows from this that the embedding of $\mathfrak{g}$ into $Vect(N_+)$ induced by the left action of G on G/B_- is equivariant.

Let $\{E_\alpha, F_\alpha\}$, $\alpha \in \Delta^+(\mathfrak{g})$ be a basis of $\mathfrak{n}_- \oplus \mathfrak{n}_+$ such that $E_\alpha \in \mathfrak{g}_\alpha, F_\alpha \in \mathfrak{g}_{-\alpha}$, and $\sigma_{\mathfrak{g}}(E_\alpha) = E_{\sigma_{\mathfrak{g}}(\alpha)}, \sigma_{\mathfrak{g}}(F_\alpha) = F_{\sigma_{\mathfrak{g}}(\alpha)}$. Let $\{\alpha_1, \cdots, \alpha_r\}$ be the set of simple roots, and set $H_{\alpha_i} = [E_{\alpha_i}, F_{\alpha_i}]$. Then $\{E_\alpha, F_\alpha, H_{\alpha_i}\}$ is a basis for $\mathfrak{g}$.

We coordinatize $N_+ \cdot B_- \cong \mathfrak{n}_+$ by $x_\alpha, \alpha \in \Delta^+(\mathfrak{g})$, with x_α dual to E_α, and LN_+ by $x_{n,\alpha}$, $n \in \mathbb{Z}, \alpha \in \Delta^+(\mathfrak{g})$, with $x_{n,\alpha}$ dual to $E_{n,\alpha}$. Then σ_{N_+} acts by sending $x_\alpha \to x_{\sigma_{\mathfrak{g}}(\alpha)}$, $\partial_{x_\alpha} \to \partial_{x_{\sigma_{\mathfrak{g}}(\alpha)}}$.

We introduce a Heisenberg algebra isomorphic to $\mathcal{A}$ with generators $\{\tilde{a}_{n,\alpha}, \tilde{a}^*_{n,\alpha}, \mathbf{1}\}$, $\alpha \in \Delta^+(\mathfrak{g})$, $n \in \mathbb{Z}$ and commutation relations

$$[\tilde{a}_{n,\alpha}, \tilde{a}_{m,\beta}] = [\tilde{a}^*_{n,\alpha}, \tilde{a}^*_{m,\beta}] = 0$$

$$[\tilde{a}_{n,\alpha}, \tilde{a}^*_{m,\beta}] = \delta_{n,-m}\delta_{\alpha,\beta}\mathbf{1}$$

This change of basis for $\mathcal{A}$ simply corresponds to the change of basis on $L\mathfrak{n}_+$ from a root basis to one on which $\sigma_{L\mathfrak{g}}$ acts diagonally. Under the identification $\tilde{a}_{n,\alpha} \to$

$\partial x_{n,\alpha}$, $\tilde{a}^*_{n,\alpha} \to x_{-n,\alpha}$, $\mathcal{A}$ corresponds to the Weyl algebra of $LN_+ \cdot LB_-$. In this basis for $\mathcal{A}$, $\sigma_M(\tilde{a}_{n,\alpha}) = \tilde{a}_{n,\sigma_{\mathfrak{g}}(\alpha)}$, $\sigma_M(\tilde{a}^*_{n,\alpha}) = \tilde{a}^*_{n,\sigma_{\mathfrak{g}}(\alpha)}$.

Similarly, we introduce a Lie algebra isomorphic to $\mathcal{H}$ with generators $\{\tilde{b}_{n,\alpha_i}, \mathbf{1}\}$ $n \in \mathbb{Z}$, $i = 1, \cdots, r$, and commutation relations

$$[\tilde{b}_{n,\alpha_i}, \tilde{b}_{m,\alpha_j}] = n(H_{\alpha_i}, H_{\alpha_j})\delta_{n,-m}\mathbf{1}$$

Here, $\sigma_\pi(\tilde{b}_{n,\alpha_i}) = \tilde{b}_{n,\sigma_{\mathfrak{g}}(\alpha_i)}$.

If $X \in \mathfrak{g}$, let $R_X(x_\alpha)$ be the polynomial vector field on $N_+ \cdot B_-$ induced by X. We write

$$R_X(x_\alpha) = \sum_{\gamma \in \Delta^+(\mathfrak{g})} R_X^\gamma(x_\alpha)\partial x_\gamma$$

We will use the notation R_X^M to denote the vector of M given by the following expression

$$\sum_{\gamma \in \Delta^+(\mathfrak{g})} R_X^\gamma(\tilde{a}^*_{0,\alpha})\tilde{a}_{-1,\gamma}|0\rangle$$

where $R_X^\gamma(\tilde{a}^*_{0,\alpha})$ means that we replace x_α by $\tilde{a}^*_{0,\alpha}$ in the polynomial R_X^γ. It follows from the untwisted Wakimoto construction ([9]) that

$$\begin{aligned}
\iota(E_{-1,\alpha_i}|0\rangle) &= R^M_{E_{\alpha_i}} \\
\iota(H_{-1,\alpha_i}|0\rangle) &= R^M_{H_{\alpha_i}} + \tilde{b}_{-1,\alpha_i} \\
\iota(F_{-1,\alpha_i}|0\rangle) &= R^M_{F_{\alpha_i}} + \lambda_{\alpha_i}\tilde{a}^*_{-1,\alpha_i} - \tilde{a}^*_{0,\alpha_i}\tilde{b}_{-1,\alpha_i}
\end{aligned}$$

where λ_α is some constant depending on the level k. It suffices to verify the lemma for these vectors, since $\{E_{\alpha_i}, F_{\alpha_i}, H_{\alpha_i}\}$ generate $\mathfrak{g}$. Because the embedding $\mathfrak{g} \to Vect(N_+ \cdot B_-)$ is equivariant, it follows that $\sigma_{M\otimes\pi}(R_X^M) = R^M_{\sigma_{\mathfrak{g}}(X)}$. Therefore the embeddings of $\iota(E_{-1,\alpha_i}|0\rangle), \iota(H_{-1,\alpha_i}|0\rangle)$ are already equivariant, and it remains to prove that $\lambda_{\alpha_i} = \lambda_{\sigma_{\mathfrak{g}}(\alpha_i)}$. To shorten notation, we denote the image of $X \in V_k(\mathfrak{g})$ under ι, $\iota(X)$, by X^ι, and suppress the vacuum vector. Let

$$S_{\alpha_i} = \sigma_{M\otimes\pi}(F^\iota_{-1,\alpha_i}) - (\sigma_{V_k(\mathfrak{g})}(F_{-1,\alpha_i}))^\iota = (\lambda_{\alpha_i} - \lambda_{\sigma_{\mathfrak{g}}(\alpha_i)})\tilde{a}^*_{-1,\sigma_{\mathfrak{g}}(\alpha_i)}$$

Using the fact that ι is a vertex algebra homomorphism, and that $Y_{M\otimes\pi}(\sigma_{M\otimes\pi}(v), z) =$

$\sigma_{M\otimes\pi} Y_{M\otimes\pi}(v,z)\sigma_{M\otimes\pi}^{-1}$, we have:

$$
\begin{aligned}
& Y_{M\otimes\pi}(\sigma_{M\otimes\pi}(E^{\iota}_{-1,\alpha_i}),z)Y_{M\otimes\pi}(S_{\alpha_i},w) \\
& \qquad \sim Y_{M\otimes\pi}(\sigma_{M\otimes\pi}(E^{\iota}_{-1,\alpha_i}),z)Y_{M\otimes\pi}(\sigma_{M\otimes\pi}(F^{\iota}_{-1,\alpha_i}),w) \\
& \qquad\quad - Y_{M\otimes\pi}(\sigma_{M\otimes\pi}(E^{\iota}_{-1,\alpha_i}),z)Y_{M\otimes\pi}((\sigma_{V_k(\mathfrak{g})}(F_{-1,\alpha_i}))^{\iota},w) \\
& \qquad \sim \sigma_{M\otimes\pi} Y_{M\otimes\pi}(E^{\iota}_{-1,\alpha_i},z),z)Y_{M\otimes\pi}(F^{\iota}_{-1,\alpha_i},w)\sigma_{M\otimes\pi}^{-1} \\
& \qquad\quad - Y_{M\otimes\pi}(E^{\iota}_{-1,\sigma_{\mathfrak{g}}(\alpha_i)},z)Y_{M\otimes\pi}(F^{\iota}_{-1,\sigma_{\mathfrak{g}}(\alpha_i)},w) \\
& \qquad \sim \frac{\sigma_{M\otimes\pi} Y_{M\otimes\pi}(H^{\iota}_{-1,\alpha_i},w)\sigma_{M\otimes\pi}^{-1}}{(z-w)} + \frac{k(E_{\alpha_i},F_{\alpha_i})}{(z-w)^2} \\
& \qquad\quad - \frac{Y_{M\otimes\pi}(H^{\iota}_{-1,\sigma_{\mathfrak{g}}(\alpha_i)},w)}{(z-w)} - \frac{k(E_{\sigma_{\mathfrak{g}}(\alpha_i)},F_{\sigma_{\mathfrak{g}}(\alpha_i)})}{(z-w)^2} \\
& \qquad \sim \frac{Y_{M\otimes\pi}(\sigma_{M\otimes\pi}(H^{\iota}_{-1,\alpha_i}),w)}{(z-w)} - \frac{Y_{M\otimes\pi}(H^{\iota}_{-1,\sigma_{\mathfrak{g}}(\alpha_i)},w)}{(z-w)} \\
& \qquad \sim 0
\end{aligned}
$$

However, we know from the Wakimoto construction that $R^{\alpha_i}_{E_{\alpha_i}} = -1$, so that $S_{\alpha_i} = 0$, which implies that $\lambda_{\alpha_i} = \lambda_{\sigma_{\mathfrak{g}}(\alpha_i)}$ as desired. This proves that ι is equivariant. The equivariance of κ is shown similarly.

$\square$

Theorem 7.5.3. *$M^\sigma \otimes \pi_\chi^{k+h^\vee,\sigma}$ has the structure of a $\hat{\mathfrak{g}}^\sigma$ representation of level k and highest weight χ.*

Proof. The homomorphism ι is σ - equivariant, so that $M^\sigma \otimes \pi_\chi^{k+h^\vee,\sigma}$ inherits a structure of σ - twisted $V_k(\mathfrak{g})$ - module. Let $J^1 \in \mathfrak{g}_i$, $J^2 \in \mathfrak{g}$. Consider the twisted fields $Y^\sigma_{M\otimes\pi}(\iota(J^l_{-1}|0\rangle),z)$, $l=1,2$. Using the associativity property for twisted modules (see [1]) and the fact that ι is a vertex algebra homomorphism, we get (the vacuum $|0\rangle$ is suppressed for brevity):

$$
\begin{aligned}
&Y^\sigma_{M\otimes\pi}(\iota(J^1_{-1}), z)Y^\sigma_{M\otimes\pi}(\iota(J^2_{-1}), w)\\
&\quad\sim \frac{z^{-i/N}w^{i/N}Y^\sigma_{M\otimes\pi}(\iota(J^1_{-1})[0]\iota(J^2_{-1}), w)}{(z-w)}\\
&\qquad + Y^\sigma_{M\otimes\pi}(\iota(J^1_{-1})[1]\iota(J^2_{-1}), w)\partial_w\frac{z^{-i/N}w^{i/N}}{(z-w)}\\
&\quad\sim \frac{z^{-i/N}w^{i/N}Y^\sigma_{M\otimes\pi}(\iota(J^1_0 J^2_{-1}), w)}{(z-w)}\\
&\qquad + Y^\sigma_{M\otimes\pi}(\iota(J^1_1 J^2_{-1}), w)\partial_w\frac{z^{-i/N}w^{i/N}}{(z-w)}\\
&\quad\sim \frac{z^{-i/N}w^{i/N}Y^\sigma_{M\otimes\pi}(\iota([J^1, J^2]_{-1}), w)}{(z-w)}\\
&\qquad + k(J_1, J_2)\partial_w\frac{z^{-i/N}w^{i/N}}{(z-w)}
\end{aligned}
$$

This implies that the fields $Y^\sigma_{M\otimes\pi}(\iota(J^l_{-1}), z)$ have the same OPE as (7.2), and hence their Fourier coefficients generate $\hat{\mathfrak{g}}^\sigma$. $\square$

We will henceforth use the notation $W^\sigma_{k,\chi}$ for $M^\sigma \otimes \pi^{k+h^\vee,\sigma}_\chi$. We call $W^\sigma_{k,\chi}$ the twisted Wakimoto module of level k and highest weight χ.

Theorem 7.5.4. *The $\hat{\mathfrak{g}}^\sigma$ module $W^\sigma_{k,\chi}$ has the same formal character as $M_{k,\chi}$ - the Verma module of level k with highest weight χ.*

Proof. The Cartan subalgebra $\mathfrak{h}^\sigma = \mathfrak{h}_0 \oplus \mathbb{C}\cdot K \oplus \mathbb{C}\cdot d$ of $\hat{\mathfrak{g}}^\sigma$ is spanned by $\{H_{0,(0,a)}, K, d\}$, $a = 1, \cdots, dim(\mathfrak{h}_0)$. When the level k is understood, we write elements $\gamma \in \mathfrak{h}^{\sigma,*}$ in the form $\bar{\gamma} + n\delta$, $n \in \frac{1}{N}\mathbb{Z}$, where $\bar{\gamma}$ is the orthogonal projection of γ to $\mathfrak{h}_0^*$, and δ is dual to d.

Unraveling the homomorphism ι, we see that $H_{0,(0,a)}$ acts on $M^\sigma \otimes \pi_\chi^{k+h^\vee,\sigma}$ as the operator

$$\sum_{\substack{(j,\alpha)\\ \alpha\in\Delta_j^+}} \sum_{n\in -\frac{j}{N}+\mathbb{Z}} \alpha(H_{0,a}) : a^*_{n,(j,\alpha)} a_{-n,(j,\alpha)} : + b_{0,(0,a)}$$

$M^\sigma \otimes \pi_\chi^{k+h^\vee,\sigma}$ has a basis of monomials of the form

$$\prod_{\substack{(j,\alpha),\alpha\in\Delta_j^+\\ n\in -\frac{j}{N}+\mathbb{Z}, n\leq 0}} (a^*_{n,(j,\alpha)})^{\lambda_{n,(j,\alpha)}} \prod_{\substack{(j,\alpha),\alpha\in\Delta_j^+\\ n\in \frac{j}{N}+\mathbb{Z}, n<0}} (a_{n,(j,\alpha)})^{\mu_{n,(j,\alpha)}} \prod_{\substack{(i,a)\\ n\in \frac{i}{N}+\mathbb{Z}, n<0}} (b_{n,(i,a)})^{\tau_{n,(i,a)}}$$

where $\lambda_{n,(j,\alpha)}$ and $\tau_{n,(i,a)}$ are non-negative integers. Now

$$\begin{aligned}
[H_{0,(0,a)}, a^*_{n,(j,\alpha)}] &= -\alpha(H_{0,(0,a)}) \\
[H_{0,(0,a)}, a_{n,(j,\alpha)}] &= \alpha(H_{0,(0,a)}) \\
[H_{0,(0,a)}, b_{n,(i,c)}] &= 0
\end{aligned}$$

An easy computation shows that the above monomial lies in the

$$\chi + \sum_{\substack{(j,\alpha),\alpha\in\Delta_j^+\\ n\in -\frac{j}{N}+\mathbb{Z}, n\leq 0}} \lambda_{n,(j,\alpha)}(-\alpha+n\delta) + \sum_{\substack{(j,\alpha),\alpha\in\Delta_j^+\\ n\in \frac{j}{N}+\mathbb{Z}, n<0}} \mu_{n,(j,\alpha)}(\alpha+n\delta) + \sum_{\substack{(i,a)\\ n\in \frac{i}{N}+\mathbb{Z}, n<0}} n\tau_{n,(i,a)}\delta$$

weight space. The formal character of $M^\sigma \otimes \pi_\chi^{k+h^\vee,\sigma}$ is therefore given by:

$$\begin{aligned}
&Ch(M^\sigma \otimes \pi_\chi^{k+h^\vee,\sigma}) = \\
&e^\chi \prod_{\substack{(j,\alpha),\alpha\in\Delta_j^+\\ n\in -\frac{j}{N}+\mathbb{Z}, n\geq 0}} \frac{1}{1-e^{-(\alpha+n\delta)}} \prod_{\substack{(j,\alpha),\alpha\in\Delta_j^+\\ n\in \frac{j}{N}+\mathbb{Z}, n>0}} \frac{1}{1-e^{-(-\alpha+n\delta)}} \prod_{\substack{(i,a)\\ n\in \frac{i}{N}+\mathbb{Z}, n>0}} \frac{1}{1-e^{-n\delta}} \qquad (7.5)\\
&= e^\chi \prod_{\beta\in\Delta_+(\hat{\mathfrak{g}}^\sigma)} \frac{1}{(1-e^{-\beta})^{mult(\beta)}}
\end{aligned}$$

which is the character of $M_{k,\chi}$ - the Verma module. $\square$

Chapter 8

Critical Level and Kac-Kazhdan conjecture

8.1 Restricted Wakimoto Modules

The $\mathcal{H}^\sigma$ module $\pi_\chi^{0,\sigma}$ has a proper submodule $\bar{\pi}_\chi^{0,\sigma}$ spanned by all non-constant monomials in $b_{n,(i,a)}$, $n < 0$, and in the notation of section 7.4.2 $\pi_\chi^{0,\sigma}/\bar{\pi}_\chi^{0,\sigma} \cong \mathbf{C}_{0,\chi}$, where we have extended $\mathbf{C}_{0,\chi}$ by 0 from $\mathcal{H}_+^\sigma \oplus \mathbb{C} \cdot \mathbf{1}$

Thus when $k = -h^\vee$, the fields $Y_{M\otimes\pi}^\sigma$ can be made to act on the smaller Fock-space

$$\overline{W}_\chi^\sigma = M^\sigma \otimes \mathbf{C}_{0,\chi}$$

We obtain

Theorem 8.1.1. $\overline{W}_\chi^\sigma$ *has the structure of a* $\hat{\mathfrak{g}}^\sigma$ *representation of highest weight* χ *and level* $-h^\vee$

We will henceforth refer to the module $\overline{W}_\chi^\sigma$ as the restricted Wakimoto module of critical level, following [9].

8.2 Restricted Contragredient Wakimoto Modules

We will also make use of the contragredient Wakimoto module $\overline{W}_\chi^{\sigma,*}$. In this section we give an explicit realization of $\overline{W}_\chi^{\sigma,*}$ in terms of a Fock space, analogous to that of

$\overline{W}^{\sigma}_{\chi}$. Let ω denote the Chevalley anti-involution of $\hat{\mathfrak{g}}^{\sigma}$. ω fixes $\mathfrak{h}_0 \oplus \mathbb{C}\cdot d \oplus \mathbb{C}\cdot K$, and maps $E_{n,(i,\alpha)}$ to a multiple of $E_{-n,(-i,-\alpha)}$.

Given an $\mathfrak{h}$ - diagonalizable category $\mathcal{O}$ module V with finite-dimensional weight spaces, we define its contragredient module V^* to be the restricted dual of V on which $\hat{\mathfrak{g}}^{\sigma}$ acts by

$$x \cdot \phi(v) = \phi(\omega(x) \cdot v)$$

where $x \in \hat{\mathfrak{g}}^{\sigma}, v \in V, \phi \in V^*$.

Let $\overline{\mathcal{A}^{\sigma}}_{+} \subset \mathcal{A}^{\sigma}$ denote the abelian subalgebra generated by $a_{n,(j,\alpha)}, a^*_{m,(k,\beta)}$, $n \leq 0, m < 0$. Let $\widetilde{M^{\sigma}} = Ind^{\mathcal{A}^{\sigma}}_{\overline{\mathcal{A}^{\sigma}}_{+}} \mathbf{C}$, where $\mathbf{C}$ denotes the trivial representation of $\overline{\mathcal{A}^{\sigma}}_{+}$. $\widetilde{M^{\sigma}}$ has a basis of monomials of the form

$$Z(\lambda,\mu) = \prod_{\substack{n_r,(j_r,\alpha_r)\\ n_r\leq 0}} (a_{n_r,(j_r,\alpha_r)})^{\lambda_{n_r,(j_r,\alpha_r)}} \prod_{\substack{n_s,(j_s,\alpha_s)\\ n_s<0}} (a^*_{n_s,(j_s,\alpha_s)})^{\mu_{n_s,(j_s,\alpha_s)}} |0\rangle$$

We define $\overline{Z(\lambda,\mu)} \in M^{\sigma}$ by

$$\overline{Z(\lambda,\mu)} = \prod_{\substack{n_r,(j_r,\alpha_r)\\ n_r\leq 0}} (a^*_{n_r,(j_r,\alpha_r)})^{\lambda_{n_r,(j_r,\alpha_r)}} \prod_{\substack{n_s,(j_s,\alpha_s)\\ n_s<0}} (a_{n_s,(j_s,\alpha_s)})^{\mu_{n_s,(j_s,\alpha_s)}} |0\rangle$$

We now define a non-degenerate bilinear pairing $\langle , \rangle : \widetilde{M^{\sigma}} \otimes M^{\sigma} \to \mathbb{C}$ by

$$\langle \overline{Z(\lambda,\mu)}, Z(\lambda,\mu)\rangle = \prod_{\substack{n_r,(j_r,\alpha_r)\\ n_r\leq 0}} \lambda_{n_r,(j_r,\alpha_r)}! \prod_{\substack{n_s,(j_s,\alpha_s)\\ n_s<0}} \mu_{n_s,(j_s,\alpha_s)}!$$

For $v \in \widetilde{M^{\sigma}}, w \in M^{\sigma}$, we have

$$\langle v, a_{n,(j,\alpha)} \cdot w\rangle = \langle a_{-n,(-j,\alpha)} v, w\rangle$$
$$\langle v, a^*_{n,(j,\alpha)} \cdot w\rangle = \langle -a^*_{-n,(-j,\alpha)} \cdot v, w\rangle$$

Let $T : End(M^{\sigma}) \to End(\widetilde{M^{\sigma}})$ denote the "transpose" map defined by

$$T(a_{n,(j,\alpha)}) = a_{-n,(-j,\alpha)}$$
$$T(a^*_{n,(j,\alpha)}) = -a^*_{-n,(-j,\alpha)}$$

and the property $T(A\cdot B) = T(B)\cdot T(A)$, $A, B \in End(M^{\sigma})$. We have

$$\langle v, A \cdot w\rangle = \langle T(A) \cdot v, w\rangle \tag{8.1}$$

It follows from (8.1) that if $x \in \hat{\mathfrak{g}}^\sigma$ then

$$\langle v, \omega(x) \cdot w \rangle = \langle T(\omega(x)) \cdot v, w \rangle$$

Thus, if we identify $\widetilde{M}^\sigma$ with the restricted dual $M^{\sigma,*}$ via the pairing $\langle , \rangle$, then the action

$$x(v) = T(\omega(x)) \cdot v$$

identifies $\widetilde{M}^\sigma$ with the $\hat{\mathfrak{g}}^\sigma$ module contragredient to $M = \overline{W}^\sigma_0$. A simple computation shows that at the level of fields, the formula for $E_{j,\alpha}(z)$ acting on $\widetilde{M}^\sigma$ is the old formula for $\omega(E_{j,\alpha})(z)$ acting on M^σ, possibly with some sign changes among the monomials. Let

$$\overline{W}^{\sigma,*}_\chi = \widetilde{M}^\sigma \otimes \mathbb{C}_{0,\chi}$$

In the same manner as in section 8.1, we extend the $\hat{\mathfrak{g}}^\sigma$ action on $\widetilde{M}^\sigma$ to $\overline{W}^{\sigma,*}_\chi$.

8.3 Proof of the Kac-Kazhdan Conjecture in the Twisted Case

In this section, let $(,)$ denote the inner product on $\mathfrak{h}^{\sigma,*}$ defined as in ([21]). At the critical level $k = -h^\vee$, a $\hat{\mathfrak{g}}^\sigma$ highest weight $\chi \in \mathfrak{h}^{\sigma,*}$ automatically satisfies the Kac-Kazhdan equation (see [22])

$$2(\chi + \rho, \beta) = n(\beta, \beta) \tag{8.2}$$

corresponding to imaginary roots $\beta \in \Delta^+_{im}(\hat{\mathfrak{g}}^\sigma)$. This implies that the Verma module M_χ is reducible. Let L_χ denote the irreducible quotient of M_χ. In [22], it is conjectured that for χ generic, i.e. such χ that do not satisfy the equation (8.2) for any real roots β, the formal character of L_χ is

$$Ch(L_\chi) = e^\chi \prod_{\alpha \in \Delta^{re}_+(\hat{\mathfrak{g}}^\sigma)} \frac{1}{(1 - e^{-\alpha})}$$

In the case of untwisted affine algebras, the Kac-Kazhdan conjecture was proved in [13]. In this section, we treat the twisted case using the same approach. The conjecture follows immediately from the following theorem:

Theorem 8.3.1. *The $\hat{\mathfrak{g}}^\sigma$ representation $\overline{W}^\sigma_\chi$ is irreducible for generic critical χ.*

8.3.1 Proof of Theorem 8.3.1

The proof of the theorem will require several steps.

There exist operators $\bar{E}_{n,(j,\alpha)}, n \in \frac{j}{N}+\mathbb{Z}, \alpha \in \Delta_j^+$, acting on $\overline{W}^\sigma_\chi$, which generate a Lie algebra isomorphic to $L\mathfrak{n}_+^\sigma$ and commute with $E_{n,(j,\alpha)}, n \in \frac{j}{N}+\mathbb{Z}, \alpha \in \Delta_j^+$. They arise from the right action of $L\mathfrak{g}^\sigma$ on the big cell of the twisted semi-infinite Flag Manifold LG^σ/LB_-^σ. Let

$$\bar{E}_{j,\alpha}(z) = \sum_{n \in \frac{j}{N}+\mathbb{Z}} \bar{E}_{n,(j,\alpha)} z^{-n-1}$$

The nilpotence of $L\mathfrak{n}_+^\sigma$ implies that the $\bar{E}_{n,(j,\alpha)}$'s are given by formulas of the form

$$\bar{E}_{j,\alpha}(z) = a_{j,\alpha}(z) + \sum_{\substack{(k,\beta) \\ 0 \le k < N, \beta \in \Delta_k^+}} a_{k,\beta}(z) P^{(k,\beta)}_{(j,\alpha)}(a^*_{(l,\gamma)}(z)) \tag{8.3}$$

where $\beta \succ \alpha$ and $\beta \succ \gamma$ in the root lattice of $\mathfrak{g}_0$, and $P^{(k,\beta)}_{(j,\alpha)}$ are polynomials.

The operators $\bar{E}_{n,(j,\alpha)}, n < 0$ generate a Lie algebra isomorphic to

$$L_-\mathfrak{n}_+^\sigma = L\mathfrak{n}_+^\sigma \cap \mathfrak{g} \otimes t^{-\frac{1}{N}}\mathbb{C}[[t^{-\frac{1}{N}}]]$$

Let F be the subspace of $\overline{W}^\sigma_\chi$ generated by $\bar{E}_{n,(j,\alpha)}, n < 0$ from the vacuum. We have a map

$$\jmath : F \otimes \mathbb{C}[a^*_{n,(j,\alpha)}]_{n \le 0} \to \overline{W}^\sigma_\chi$$

sending a vector of the form

$$\bar{E}_{n_1,(j_1,\alpha_1)} \cdots \bar{E}_{n_k,(j_k,\alpha_r)}|0\rangle \otimes a^*_{m_1,(k_1,\beta_1)} \cdots a^*_{m_s,(k_s,\beta_s)}|0\rangle$$

to the element of $\overline{W}^\sigma_\chi$ given by

$$\bar{E}_{n_1,(j_1,\alpha_1)} \cdots \bar{E}_{n_k,(j_k,\alpha_r)} a^*_{m_1,(k_1,\beta_1)} \cdots a^*_{m_s,(k_s,\beta_s)}|0\rangle$$

Lemma 8.3.1. *The map $\jmath$ is an isomorphism*

Proof. It is clear that $\jmath$ is injective. The expression (8.3) for $\bar{E}_{j,\alpha}(z)$ implies that we can solve for $a_{j,\alpha}(z)$ in terms of $a^*_{k,\beta}(z)$ and $\bar{E}_{i,\gamma}(z)$. In other words,

$$a_{j,\alpha}(z) = \bar{E}_{j,\alpha}(z) + \sum \bar{E}_{k,\beta}(z) R^{(k,\beta)}_{(j,\alpha)}(a^*_{i,\gamma}(z))$$

where in the summation, $\beta \succ \alpha$ in the root lattice of $\mathfrak{g}_0$, and $R^{(k,\beta)}_{(j,\alpha)}$ are polynomials. Now, $\overline{W}^\sigma_\chi$ has a basis of monomials of the form

$$a_{n_1,(j_1,\alpha_1)} \cdots a_{n_r,(j_r,\alpha_s)} a^*_{m_1,(k_1,\beta_1)} \cdots a^*_{m_s,(k_s,\beta_s)}|0\rangle$$

where $n_p < 0, m_q \leq 0$. Thus we see that $\jmath$ is surjective.

□

We use the map $\jmath$ to identify $F \otimes \mathbb{C}[a^*_{n,(j,\alpha)}]_{n\leq 0}$ and $\overline{W}^\sigma_\chi$. The operators $\bar{E}_{n,(j,\alpha)}$ commute with the action of $L\mathfrak{n}^\sigma_+ \in \hat{\mathfrak{g}}^\sigma$, and so under this identification, if $P \in F, Q \in \mathbb{C}[a^*_{n,(j,\alpha)}]_{n\leq 0}$

$$E_{m,(k,\beta)} \cdot (P \otimes Q) = P \otimes (E_{m,(k,\beta)} \cdot Q)$$

Let

$$L_+\mathfrak{n}^\sigma_+ = L\mathfrak{n}^\sigma_+ \cap \mathfrak{g} \otimes t^{\frac{1}{N}}\mathbb{C}[[t^{\frac{1}{N}}]] \subset \hat{\mathfrak{g}}^\sigma$$

Then $L_+\mathfrak{n}^\sigma_+$ acts co-freely on $\mathbb{C}[a^*_{n,(j,\alpha)}]_{n\leq 0}$, i.e. given $X \in \mathfrak{U}(L_+\mathfrak{n}^\sigma_+)$ (here $\mathfrak{U}()$ stands for enveloping algebra), there is a unique $Q \in \mathbb{C}[a^*_{n,(j,\alpha)}]_{n\leq 0}$ such that $X \cdot Q = |0\rangle$. Now suppose that $v \in \overline{W}^\sigma_\chi$ is a singular vector. Then in particular, it must be killed by $L_+\mathfrak{n}^\sigma_+$, and so must be of the form $P \otimes |0\rangle, P \in F$. Thus all potential singular vectors must lie in the subspace $F \otimes |0\rangle \subset F \otimes \mathbb{C}[a^*_{n,(j,\alpha)}]_{n\leq 0}$.

The module $\overline{W}^\sigma_\chi$ is graded by $\Lambda(\hat{\mathfrak{g}}^\sigma)$ (if $v \in \overline{W}^\sigma_\chi$ belongs to the μ weight space, $deg(v) = \chi - \mu$). Using the notation of Theorem 7.5.4, we write $deg(v) = \beta = \bar{\beta} + m\delta$, where $\beta \in \Lambda(\mathfrak{g}), m \in \frac{1}{N}\mathbb{Z}$.

Lemma 8.3.2. *If $v \in F \otimes |0\rangle$, $deg(v) = \beta$, then $\bar{\beta} \neq 0$*

Proof. The degree of the operator $\bar{E}_{n,(j,\alpha)}, n < 0, \alpha \in \Delta^+(j)$ is $\alpha + n\delta$. It follows that repeated application of such operators to the vacuum can never result in a vector with degree $m\delta$. □

We can apply the same line of argument to the contragredient module $\overline{W}_{\chi}^{\sigma,*}$. The operators $E_{n,(j,-\alpha)}$ $n > 0, \alpha \in \Delta_j^+$ generate the Lie algebra

$$L_+ n_-^\sigma = \mathfrak{n}_- \otimes t^{\frac{1}{N}}\mathbb{C}[[t^{\frac{1}{N}}]] \subset \hat{\mathfrak{g}}^\sigma$$

The operators $\bar{E}_{n,(j,\alpha)}$ act on $\overline{W}_{\chi}^{\sigma,*}$ as before, and commute with the action of $L_+\mathfrak{n}_-^\sigma$. Let $\bar{F}$ denote the subspace generated by $\bar{E}_{n,(j,\alpha)}$, $n \leq 0$ from the vacuum in $\overline{W}_{\chi}^{\sigma,*}$. We can show as before that $\overline{W}_{\chi}^{\sigma,*} = \bar{F} \otimes \mathbb{C}[a^*_{n,(j,\alpha)}]_{n<0}$, that if $E_{n,(j,\alpha)} \in L_+\mathfrak{n}_-^\sigma, P \in \bar{F}, Q \in \mathbb{C}[a^*_{n,(j,\alpha)}]_{n<0}$, then

$$E_{n,(j,\alpha)} \cdot (P \otimes Q) = P \otimes (E_{n,(j,\alpha)} \cdot Q)$$

and that the action of $L_+\mathfrak{n}_-^\sigma$ on $\mathbb{C}[a^*_{n,(j,\alpha)}]_{n<0}$ is co-free. Thus, any singular vector $v \in \overline{W}_{\chi}^{\sigma,*}$ lies in $\bar{F} \otimes |0\rangle$, and can be shown to have degree $deg(v) = \beta$, $\bar{\beta} \neq 0$.

We are now ready to prove Theorem 8.3.1

Proof. The formal character $Ch(V)$ of a category $\mathcal{O}$ $\hat{\mathfrak{g}}^\sigma$ module V can be uniquely expressed as a linear combination of characters of irreducible highest-weight modules $Ch(L_\psi), \psi \in \mathfrak{h}^{\sigma,*}$. Let $C_V^\psi \in \mathbb{Z}$ denote the coefficient with which $Ch(L_\psi)$ occurs in $Ch(V)$.

Suppose that $\overline{W}_{\chi}^{\sigma}$ has a proper submodule. Then either $\overline{W}_{\chi}^{\sigma}$ or $\overline{W}_{\chi}^{\sigma,*}$ has a singular vector $v \neq |0\rangle$. Suppose first that $v \in \overline{W}_{\chi}^{\sigma}$, and that v has weight μ. It follows that $C^\mu_{\overline{W}_{\chi}^{\sigma}} > 0$. Now, $\overline{W}_{\chi}^{\sigma}$ is a quotient of $W^\sigma_{-h^\vee,\chi} = M^\sigma \otimes \pi_\chi^{0,\sigma}$, which by Theorem 7.5.4 has the same character as the Verma module $M_{-h^\vee,\chi}$. It follows that $C^\mu_{M_{-h^\vee,\chi}} > 0$. By Lemma 8.3.2, $\chi - \mu = \beta$, $\bar{\beta} \neq 0$. Proposition (4.1) from [22] then implies that $M_{-h^\vee,\chi}$ contains a singular vector of weight μ, and hence degree $\bar{\beta} + m\delta$, contradicting the following Lemma.

The case where $v \in \overline{W}_{\chi}^{\sigma,*}$ is treated in the same way. □

Lemma 8.3.3. *Let $\chi \in \mathfrak{h}^{\sigma,*}$ be a generic critical highest weight, and $v \in M_{-h^\vee,\chi}$ a singular vector. Then $deg(v) = m\delta, m \in \frac{1}{N}\mathbb{Z}$*

Proof. Suppose that v has weight μ. By Theorem 2 in [22], there exists a sequence of positive roots $\gamma_1, \cdots, \gamma_k \in \Delta^+(\hat{\mathfrak{g}}^\sigma)$, and a sequence of positive integers $n_i, \cdots, n_k$ such that

$$2(\chi + \rho - n_1\gamma_1 - \cdots - n_{i-1}\gamma_{i-1}, \gamma_i) = n_i(\gamma_i, \gamma_i)$$

for $i = 1, \cdots, k$ and $\chi - \mu = \sum_{i=1}^k n_i\gamma_i$. Now $\chi - \mu = \bar{\beta} + m\delta$, $\bar{\beta} \neq 0$, so $\gamma_r \in \Delta^+_{re}(\hat{\mathfrak{g}}^\sigma)$ for some r. Let us choose the smallest such r. We have that

$$2(\chi + \rho - n_1\gamma_1 - \cdots - n_{r-1}\gamma_{r-1}, \gamma_r) = n_r(\gamma_r, \gamma_r)$$

and since real and imaginary roots are orthogonal, $2(\chi + \rho, \gamma_r) = n_r(\gamma_r, \gamma_r)$, contradicting the assumption that χ is generic. $\square$

Chapter 9

Examples

9.1 Wakimoto Realization of $A_2^{(2)}$

In this section we give explicit realizations of the twisted affine algebra $A_2^{(2)}$. Let $R_{i,j}$ denote the 3×3 matrix with a 1 in position (i, j).

Here $\mathfrak{g} = \mathfrak{sl}_3$, where we take the standard realization as traceless $3x3$ matrices. Under σ, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathfrak{sl}_2$. As a $\mathfrak{g}_0$ - representation, $\mathfrak{g}_1$ is isomorphic to the irreducible 5 - dimensional representation. Let α denote the highest root of $\mathfrak{sl}_2$. We choose a basis as follows

$$
\begin{aligned}
E_{0,\alpha} &= R_{(1,1)} + R_{(2,3)} \\
E_{1,\alpha} &= R_{(1,1)} - R_{(2,3)} \\
E_{1,2\alpha} &= -2R_{(1,3)} \\
H_{0,1} &= R_{(1,1)} - R_{(3,3)} \\
H_{1,1} &= R_{(1,1)} - 2R_{(2,2)} + R_{(3,3)} \\
E_{0,-\alpha} &= R_{(2,1)} + R_{(3,2)} \\
E_{1,-\alpha} &= R_{(2,1)} - R_{(3,2)} \\
E_{1,-2\alpha} &= -2R_{(3,1)}
\end{aligned}
$$

The Wakimoto realization in this case looks as follows (we omit the fields $E_{1,2\alpha}(z)$, $E_{1,-2\alpha}(z)$ as these can be calculated via the OPE from the other generators):

$$
\begin{aligned}
E_{(0,\alpha)}(z) &= -a_{(0,\alpha)}(z) - 1/2 : a^*_{(1,\alpha)}(z)a_{(1,2\alpha)}(z) : \\
E_{(1,\alpha)}(z) &= -a_{(1,\alpha)}(z) + 1/2 : a^*_{(0,\alpha)}(z)a_{(1,2\alpha)}(z) : \\
H_{(0,1)}(z) &= - : a^*_{(0,\alpha)}(z)a_{(0,\alpha)}(z) : - : a^*_{(1,\alpha)}(z)a_{(1,\alpha)}(z) : -2 : a^*_{(1,2\alpha)}(z)a_{(1,2\alpha)}(z) : +b_{(0,1)}(z) \\
H_{(1,1)}(z) &= -3 : a^*_{(0,\alpha)}(z)a_{(0,\alpha)}(z) : -3 : a^*_{(1,\alpha)}(z)a_{(0,\alpha)}(z) : +b_{(1,1)}(z) \\
E_{(0,-\alpha)}(z) &= 1/2 : (a^*_{(0,\alpha)}(z))^2 a_{(0,\alpha)}(z) : +3/2 : (a^*_{(1,\alpha)}(z))^2 a_{(0,\alpha)}(z) : \\
&\quad + 2 : a^*_{(0,\alpha)}(z)a^*_{(1,\alpha)}(z)a_{(1,\alpha)}(z) : -2 : a^*_{(1,2\alpha)}(z)a_{(1,\alpha)}(z) : + \\
&\quad 1/4 : a^*_{(1,\alpha)}(z)(a^*_{(0,\alpha)}(z))^2 a_{(1,2\alpha)}(z) : -1/4 : (a^*_{(1,\alpha)}(z))^3 a_{(1,2\alpha)}(z) : + \\
&\quad : a^*_{(0,\alpha)}(z)a^*_{(1,2\alpha)}(z)a_{(1,2\alpha)}(z) : +(-1-2k)\partial_z a^*_{0,\alpha}(z) - a^*_{0,\alpha}(z)b_{(0,1)}(z) \\
E_{(1,-\alpha)}(z) &= 2 : a^*_{(1,2\alpha)}(z)a_{(0,\alpha)}(z) : +2 : a^*_{(0,\alpha)}(z)a^*_{(1,\alpha)}(z)a_{(0,\alpha)}(z) : \\
&\quad + 3/2 : (a^*_{(0,\alpha)}(z))^2 a_{(1,\alpha)}(z) : +1/2 : (a^*_{(1,\alpha)}(z))^2 a_{(1,\alpha)}(z) : \\
&\quad + 1/4 : (a^*_{(0,\alpha)}(z))^3 a_{(1,2\alpha)}(z) : -1/4 : a^*_{(0,\alpha)}(z)(a^*_{(1,\alpha)}(z))^2 a_{(1,2\alpha)}(z) : \\
&\quad + : a^*_{(1,\alpha)}(z)a^*_{(1,2\alpha)}(z)a_{(1,2\alpha)}(z) : +(-1-2k)\partial_z a^*_{1,\alpha}(z) - a^*_{1,\alpha}(z)b_{(1,1)}(z)
\end{aligned}
$$

Remark: After this work was completed, we were informed by Prof. T. Takebe that some of our results had been obtained independently by Prof. G. Kuroki.

Appendix A

Formal Discs

In this paper, we make frequent use of "formal geometry", such as (algebraic) discs, punctured discs, and bundles on these. The purpose of this appendix is to make precise what is meant by these concepts.

Definition A.0.1. Let X be a smooth algebraic curve over $\mathbb{C}$, and $x \in X$. By a *disc* we will mean a scheme isomorphic to $\mathcal{D}_x = Spec(\mathcal{O}_x)$, where $\mathcal{O}_x$ is the completion of the local ring at x. By a *punctured disc* we will understand a scheme isomorphic to $\mathcal{D}_x^\times = Spec(\mathcal{K}_x)$, where $\mathcal{K}_x$ is the field of fractions of $\mathcal{O}_x$.

Remark 11. Note that a formal disc is here taken to be a *scheme*, not a *formal scheme*.

Choosing a formal coordinate z at x, i.e. a generator of the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_x$, identifies $\mathcal{D}_x$ with $Spec(\mathbb{C}[[z]])$, and $\mathcal{D}_x^\times$ with $\mathbb{C}((z))$. There is a map $\mathcal{D}_x \mapsto X$, but $\mathcal{D}_x$ is neither closed nor open in X.

We will also be considering the restriction to $\mathcal{D}_x$ of a vector bundle $\mathcal{V}$ defined on X. By this we mean the free $\mathcal{O}_x$–module $\mathcal{V}(\mathcal{D}_x)$ which is the completion of the $\mathbb{C}[\Sigma]$–module $\Gamma(\Sigma, \mathcal{V})$ at x, where Σ is a Zariski open affine neighborhood of x (this notion is independent of the affine Σ chosen). By the restriction of $\mathcal{V}$ to $\mathcal{D}_x^\times$ we understand the free $\mathcal{K}_x$–module $\mathcal{V}(\mathcal{D}_x^\times) = \mathcal{V}(\mathcal{D}_x) \otimes_{\mathcal{O}_x} \mathcal{K}_x$.

Remark 12. It is clear from the above that if $\mathcal{V}$ carries an equivariant structure on X, and x is a fixed point, this structure restricts to $\mathcal{D}_x$ and $\mathcal{D}_x^\times$.

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