ANSWER KEY

MA 123 – Fall 2024
Midterm Examination 1
6:30–8:30 pm Thursday, October 17, 2024

1. (3 points)

First Name : ANSWER
Last (Family) Name : KEY
BU ID number : Y'all Smart

Directions

All of your work must be shown in this exam booklet.

Backpacks and bags of any type must be stored in the front of the room. Keep your BU ID with you.

Phones (of any type) must be turned OFF and in your backpack (silent mode is not permitted).

Books, notes, extra papers are not permitted.

The use of any electronic, mechanical, photonic, or quantum device to carry out any calculations or any steps in any part of the solutions is not permitted.

Please do not separate the pages of this exam booklet.

Please box or circle your final answers.

If you have a question about a problem, please raise your hand and a proctor will come to your seat to answer it.

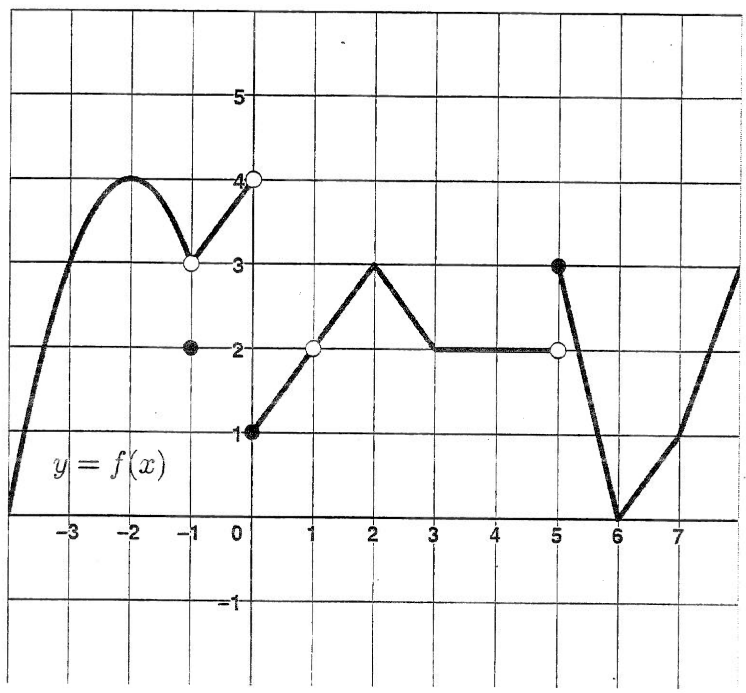
Answers that are written logically and clearly will receive higher scores.

There are nine calculus problems on the exam and one logistical problem that simply asks for your first name, last (family) name, and BU ID number. The logistical problem is problem 1, at the top of this page. The calculus problems are numbered 2–10, starting on the second page of this exam booklet.

The entire exam booklet consists of 10 (ten) pages, double-sided, including this cover page. Please make sure that your exam booklet includes all 5 pieces of paper.

For your scan, please make sure you scan all ten pages, including this cover page.

2. (7 points: 1 for each part) Here is the graph $y = f(x)$ of a function $f(x)$. Consider all $x \in [-4, 8]$.



For each of the following limits, place your answer in the box. If the limit is a number, put the number in the box. If the limit is $+\infty$ or $-\infty$, write that in the box. If the limit does not exist and is not $\pm\infty$, then write DNE in the box.

a. $\lim_{x\to 0} f(x) = \boxed{\text{DNE}}$ because $\lim_{x\to 0^-} f(x) = 4$ and $\lim_{x\to 0^+} f(x) = 1$.

b. $\lim_{x\to 5^-} f(x) = \boxed{2}$

c. $\lim_{x\to 5^+} f(x) = \boxed{3}$

For each of the following derivatives, place your answer in the box. If the derivative exists, put the value of the derivative in the box. If the derivative does not exist, write DNE in the box.

d. $f'(-2) = \boxed{0}$

e. $f'(1) = \boxed{\text{DNE}}$ because $f(x)$ is not defined @ $x = 1$, open circle.

f. $f'(6) = \boxed{\text{DNE}}$ because $f(x)$ has a corner @ $x = 6$.

In the final box, list the values of x where $f(x)$ is discontinuous:

g. $\boxed{-1, 0, 1, 5}$

$S(8)=256$ m

3. (12 points: 3 for each part) In this problem, you are to consider the function

$$s(t) = -4t^2 + 64t. \qquad S'(t) = -8t+64$$

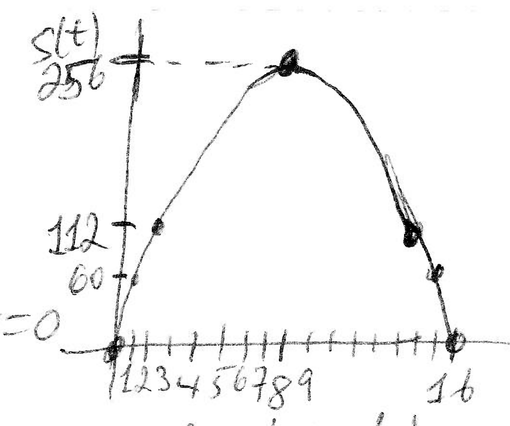
It gives the height (measured in meters) of a rock, thrown vertically straight up from the ground at time $t = 0$. (Here, we are on a planet that is smaller than Earth, so the acceleration due to gravity is weaker than on Earth.) Time ($t \geq 0$) is measured in seconds. For each of the following four parts, you must include **UNITS** in your answer and show enough work to justify your answer. You may use any method that is appropriate from Chapters 2.1–2.6 and 3.1–3.9.

a. Find the average velocity of the rock on the time interval $[1, 2]$.

On $[1,2]$, $V_{AVE} = \dfrac{s(2) - s(1)}{2-1} = \dfrac{112-60}{1} = \boxed{52 \frac{m}{sec}}$

b. Find the instantaneous velocity of the rock at time $t = 2$.

@ $t=2$, $V_{INSTAN} = S'(2) = -16+64 = \boxed{48 \frac{m}{sec}}$



c. Find the time at which the rock is at its highest point.

@ its highest point, the rock has zero velocity, $v(t) = S'(t) = 0 \Rightarrow -8t+64=0$

So, answer is $\boxed{t=8 \text{ sec}}$

graph not needed, but useful for thinking about the problem.

d. Find the slope of the tangent line to the graph of $y = s(t)$ at $t = 5$.

$M_{tan} = S'(5) = -40+64 = \boxed{24}$

d) no need for units but, do not penalize if a student gives units

4. **(16 points: 4 for each part)**
For each of the following four limits, if it exists, evaluate it using only techniques learned so far in class this semester. If the limit is infinite, indicate if it is $+\infty$ or $-\infty$. If the limit does not exist and is NOT infinite, write DNE. Show enough work to justify your answer.

a. $\lim_{x\to 3} \frac{x^2 - x - 6}{x^2 + x - 12} = \boxed{\frac{5}{7}}$

OK to cancel $\frac{(x-3)}{(x-3)}$ because we do not evaluate @ $x=3$.

$$\lim_{x\to 3} \frac{x^2-x-6}{x^2+x-12} = \lim_{x\to 3} \frac{\cancel{(x-3)}(x+2)}{\cancel{(x-3)}(x+4)} = \lim_{x\to 3} \frac{x+2}{x+4} = \frac{5}{7} \checkmark$$

b. $\lim_{x\to -\infty} \frac{2x^2 - x + 8}{5x^2 + 7} = \boxed{\frac{2}{5}}$

can state: highest powers same in numerator & denom.

$$\lim_{x\to -\infty} \frac{2x^2-x+8}{5x^2+7} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to -\infty} \frac{2-\frac{1}{x}+\frac{8}{x^2}}{5+\frac{7}{x^2}} = \frac{2}{5} \checkmark$$

c. $\lim_{x\to 0} \frac{\sin(4x)}{x} = \boxed{4}$

$\theta = 4x$

$$\lim_{x\to 0} \frac{\sin(4x)}{x} \cdot \frac{4}{4} = \lim_{x\to 0} \frac{4\sin(4x)}{4x} = 4 \cdot \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 4$$

(OK to use $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$)

d. $\lim_{h\to 0} \frac{\cos(h) - 1}{h} = \boxed{0}$

limit def. of derivative

Let $f(x) = \cos(x)$, then $f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\cos(h)-1}{h}$

& hence, we simply calculate $f'(0) = \frac{d}{dx}\cos x \Big|_{x=0} = -\sin(0) = 0 \checkmark$

not defined @ $x=3,4$ because denom $=0$ there.

5. (8 points: 4 for each part)

a. Find all of the vertical asymptotes of the following function: $g(x) = \dfrac{x-3}{x^2-7x+12}$
Show the work needed to derive the answer.

$$g(x) = \frac{\cancel{(x-3)}}{\cancel{(x-3)}(x-4)} = \frac{1}{(x-4)} \quad \text{for all } x \neq 3,4.$$

$$\Rightarrow \lim_{x\to 4^-} g(x) = \lim_{x\to 4^-} \frac{1}{(x-4)} = -\infty \Rightarrow \boxed{x=4 \text{ is a VA of } g}$$

also $\lim_{x\to 4^+} g(x) = \lim_{x\to 4^+} \frac{1}{(x-4)} = +\infty$ (either calc. is enough!)

[but, $\lim_{x\to 3^+} g(x) = \lim_{x\to 3^+} \frac{1}{(x-4)} = -1 \Rightarrow$ NO VA @ $x=3$]

b. Write down all of the INTERVALS on which the following function $f(x)$ is continuous:
$f(x) = (x^2-2x+4)^{2/3}$
Show the work needed to obtain the answer.

$\boxed{(-\infty, \infty)}$ ← answer

x^2-2x+4 is a quadratic (polynomial)
& Ch2.6 ⇒ all polynomials are continuous for all x.

cube roots, like $(x^2-2x+4)^{1/3}$ are defined & continuous for all values of x (think also of $x^{1/3}$, for example)

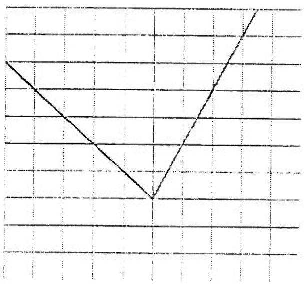
& finally, squaring is a continuous function for all x

So, $\left[(x^2-2x+4)^{1/3}\right]^2$ is continuous for all x
i.e. for all $x \in (-\infty, \infty)$

here, $f(x)$ is the composition of three continuous functions.

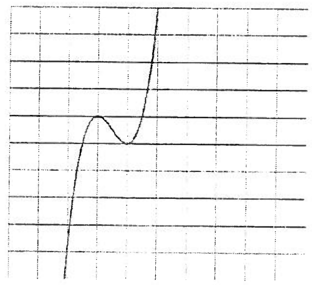
6. **(9 points: 3 for each part)** For each function that is graphed in parts (a)-(c), write in the box below it the number of the graph (i)-(ix) that is the graph of its derivative. The graph in each box shows $x \in [-5, 5]$ and $y \in [-5, 5]$, and the origin $(x, y) = (0, 0)$ is at the center of the box.

(a) graph of f



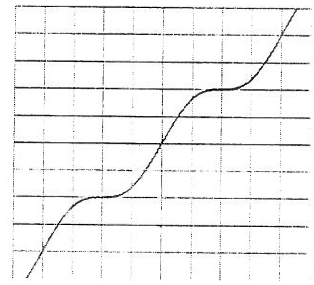
$\boxed{\text{iv}}$

(b) graph of g



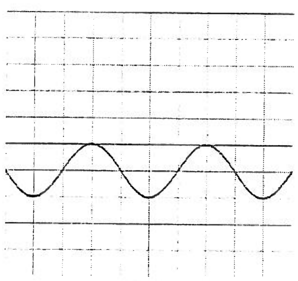
$\boxed{\text{ii}}$

(c) graph of h

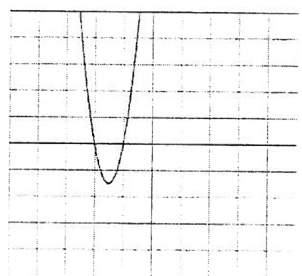


$\boxed{\text{V}}$

(i)



(ii)



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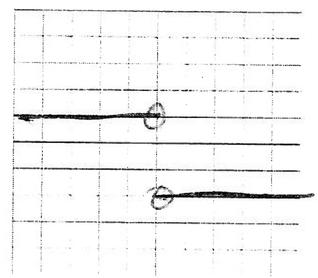

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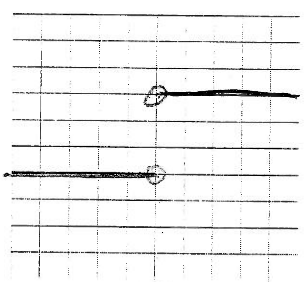
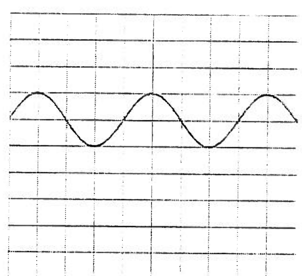
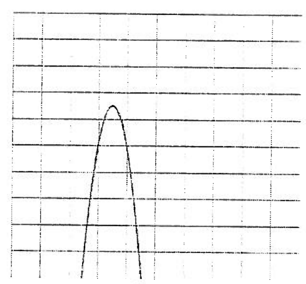


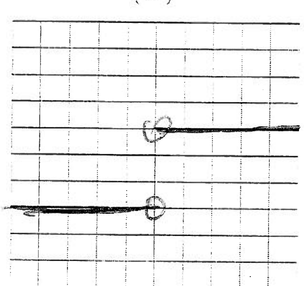
(v)



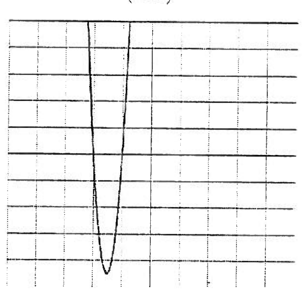
(vi)



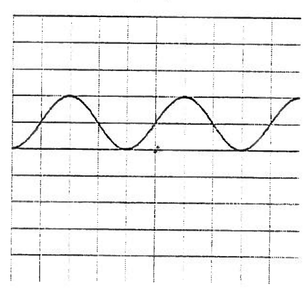
(vii)



(viii)



(ix)



7. **(12 points)** In this problem, you are to consider the function

$$f(x) = \frac{2}{x+3}$$

It is defined for all $x < -3$ and for all $x > -3$. Using only the definition of the derivative and the limit theorems learned so far this semester in the class, calculate $f'(x)$ for all x where f is defined. You will not receive any credit if you do not start with the limit definition of $f'(x)$ or if you use rules for derivatives without calculating a limit.

general limit definition of deriv: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

here, $f(x) = \frac{2}{x+3}$

so, $f'(x) = \lim_{h \to 0} \left[\frac{\frac{2}{x+h+3} - \frac{2}{x+3}}{h} \right]$ put both terms in the numerator over a common denominator

$\Rightarrow f'(x) = \lim_{h \to 0} \left[\frac{2(x+3) - 2(x+h+3)}{h(x+3)(x+h+3)} \right]$ now simplify the numerator

$\Rightarrow f'(x) = \lim_{h \to 0} \left[\frac{-2\cancel{h}}{\cancel{h}(x+3)(x+h+3)} \right]$

$\Rightarrow \boxed{f'(x) = \frac{-2}{(x+3)^2}}$

(which one may check using the power rule)

PLEASE DO NOT PENALIZE FOR SIMPLIFYING

8. (12 points: 3 for each part) In each part, calculate the derivative $\frac{dy}{dx}$ by any appropriate method that we have learned so far. Please do **NOT** simplify your answers.

a. $y = 2^x + e^{3x}$

$$\boxed{y' = 2^x \ln 2 + 3e^{3x}}$$

(recall $\frac{d}{dx} b^x = b^x (\ln b)$.)

b. $y = \dfrac{\sin(7x)}{x^2+4}$

$$\boxed{y' = \frac{(x^2+4)\, 7\cos(7x) - \sin(7x)\cdot 2x}{(x^2+4)^2}}$$

(quotient rule)

c. $y = \sec\left(\sin(x^2)\right)$

$$\boxed{y' = \sec(\sin(x^2))\tan(\sin(x^2))\cdot \cos(x^2)\cdot 2x}$$

(chain rule)

d. $y = e^4 + \ln(\cos^2(x))$

$$\boxed{y' = 0 + \frac{1}{\cos^2 x}\cdot 2\cos x\,(-\sin x)} \Rightarrow y' = -2\tan x$$

OK, too

derivative of a constant is zero

chain rule

outer function is $f(u) = \ln u \Rightarrow f'(u) = \frac{1}{u}$

inner function is $g(x) = \cos^2(x) \Rightarrow g'(x) = -2\cos(x)\sin(x)$ by chain rule.

9. **(9 points)** In this problem, you are working with the equation $4x\sqrt{y} + \frac{y^2}{3} = 63$. Calculate an equation of the tangent line to the curve $y(x)$ at the point $(x, y) = (3, 9)$. You may write your answer either in slope-intercept form or in point-slope form. You must show all of your work.

given $4x\sqrt{y} + \frac{y^2}{3} = 63$

✓ (3,9) is on the curve because $4(3)\sqrt{9} + \frac{9^2}{3} = 63$ ✓

use implicit differentiation

$$\frac{d}{dx} \Rightarrow 4\sqrt{y} + \frac{2x}{\sqrt{y}} y' + \frac{2}{3} y y' = 0$$

$$\Rightarrow \left(\frac{2x}{\sqrt{y}} + \frac{2}{3}y\right) y' = -4\sqrt{y}$$

$$\Rightarrow y' = \frac{-4\sqrt{y}}{\left(\frac{2x}{\sqrt{y}} + \frac{2}{3}y\right)}$$

$$\Rightarrow @ (3,9),\ y' = \frac{-12}{(2+6)} = \boxed{\frac{-3}{2} = m_{tan}}$$

TANGENT LINE

Pt.-slope form: $\frac{y-9}{x-3} = \frac{-3}{2}$ (or $y - 9 = \frac{-3}{2}(x-3)$)

Slope-intercept form: $y = \frac{-3}{2}x + \frac{27}{2}$

✓ (3,9) is also on the tangent line ✓

10. **(12 points: 3 for each part)** In this problem, you are given the following table of data about two functions $f(x)$ and $g(x)$ and their derivatives $f'(x)$ and $g'(x)$ at the values of x indicated across the top row of the table.

	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$
$f(x)$	0	3	-6	2	4
$f'(x)$	3	7	5	10	-8
$g(x)$	3	4	2	1	3
$g'(x)$	2	-3	4	-12	-2

In each part, calculate the indicated derivative.

product rule

a. $u'(4)$ where $u(x) = x^2 f(x)$ $\Rightarrow u' = 2x f(x) + x^2 f'(x)$

$\Rightarrow u'(4) = 8f(4) + 16f'(4) = 16 + 160 = \boxed{176}$

b. $v'(3)$ where $v(x) = f(g(x))$ $\Rightarrow v'(x) = f'(g(x))g'(x)$ by Chain Rule

$\Rightarrow v'(3) = f'(g(3))g'(3) = f'(2)\cdot 4 = \boxed{28}$

c. $w'(5)$ where $w(x) = \frac{f(x)}{g(x)}$ $\Rightarrow w'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$ by quotient rule.

$\Rightarrow w'(5) = \dfrac{f'(5)g(5) - f(5)g'(5)}{(g(5))^2} = \dfrac{(-8)(3) - (4)(-2)}{(3)^2} = \boxed{\dfrac{-16}{9}}$

d. $z'(2)$ where $z(x) = g(f(x))$

$\Rightarrow z'(x) = g'(f(x))f'(x)$ by chain rule

$\Rightarrow z'(2) = g'(f(2))f'(2) = g'(3)\cdot 7 = \boxed{28}$