SOLUTIONS: PRACTICE TEST – MIDTERM 2

MA123 A1–A4

1. First name: Student
   Family name: O'Calculus
   BUID #: $e^{2\pi i}-1=0$

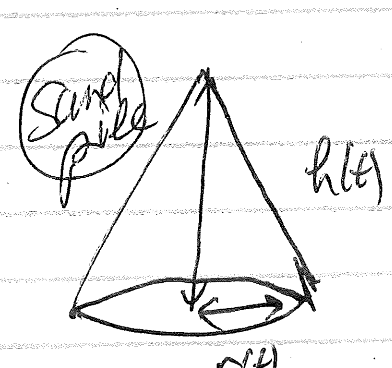
2. a) $f(x)=\pi^5+\sin^{-1}(e^{-3x})$

$$\Rightarrow f'(x)=0+\frac{1}{\sqrt{1-(e^{-3x})^2}}\left(-3e^{-3x}\right)$$

b) $g(x)=\tan^{-1}(e^{\cos x})$

$$\Rightarrow g'(x)=\frac{1}{1+(e^{\cos x})^2}\left(-\sin x\, e^{\cos x}\right)$$

c) $h(x)=\cos^{-1}(x) \Rightarrow h'(x)=\frac{-1}{\sqrt{1-x^2}}$

3) 

given $\frac{dh}{dt}=2\ \frac{\text{inches}}{\text{second}}$

$V=\frac{\pi}{3}r^2h$

& $r(t)=3h(t)$

SOLUTIONS

3) contd.

use $r = 3h \Rightarrow V = \frac{\pi}{3}(3h)^2 h = 3\pi h^3$

differentiate $\Rightarrow \frac{dV}{dt} = 9\pi h^2 \frac{dh}{dt}$

So, when the pile is 10 inches high, we have

$$\boxed{\frac{dV}{dt} = 9\pi(10)^2(2) = 1{,}800\pi \ \frac{\text{inches}^3}{\text{second}}}$$

4) $f(x) = x^4 - 4x^3 + 4x^2$ on $[-1, 3]$

$\Rightarrow f'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2)$

$\Rightarrow \boxed{f'(x) = 4x(x-2)(x-1)}$

a) $x = 0, 1, 2$ are the three critical points

b) $x = 0$ and $x = 2$ are local minima
(use either first-deriv. test or 2nd deriv. test)

c) $x = 1$ is a local max

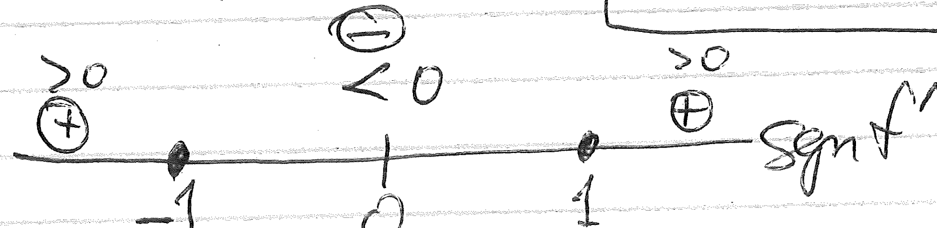
$\ominus$ 0 $\oplus$ 1 $\ominus$ 2 $\oplus$ sgn f'

4) d) $f(0)=0$ & $f(2)=0$ $\Rightarrow$ $\boxed{x=0 \text{ \& } x=2 \text{ are absolute min.}}$

4) e) $f(1)=1$
$f(-1)=9$
$f(3)=9$
$\Rightarrow$ $\boxed{x=-1 \text{ and } x=3 \text{ are absolute max of } f \text{ on } [-1,3]}$

5) $f(x)=x-2\tan^{-1}x$, $f'=1-\dfrac{2}{1+x^2}$

a) f' exists for all x
& $f'=0$ @ $x=\pm 1$ $\Rightarrow$ $\boxed{x=\pm 1 \text{ are the two critical points}}$

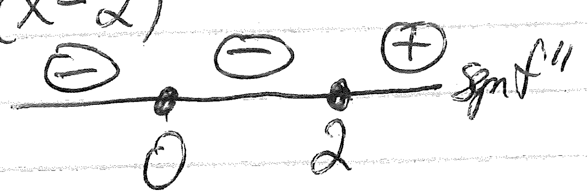
b)


$\Rightarrow$ $\boxed{f \uparrow\text{ing on } (-\infty,-1) \text{ \& on } (1,\infty)}$ because $f'(x)>0$ there

c) $\boxed{f \downarrow\text{ing on } (-1,1)}$
b/c $f'(x)<0$ there

6) $f(x)=3x^5-10x^4$, $f'(x)=15x^4-40x^3$

a) $f''(x)=60x^3-120x^2=60x^2(x-2)$



b) $\boxed{f(x) \text{ concave up on } (2,\infty)}$
(where $f''(x)>0$)

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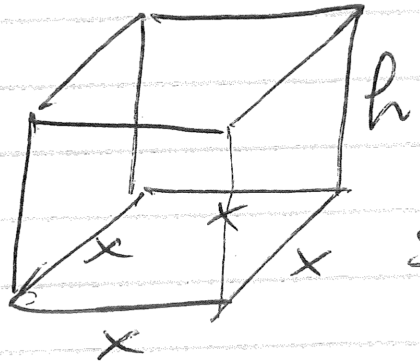
6) c) $\boxed{f(x) \text{ concave down on } (-\infty, 0) \text{ \& on } (0,2)}$

because $f''(x) < 0$ there

6) d) $\boxed{x=2 \text{ is the only inflection point of } f(x)}$

7)



square base (of sides x & x)

area of top $= x^2$

area of bottom $= x^2$

area of 4 sides $= 4xh$

$V = x^2 h = 54 \ m^3$

solve constraint to find one variable as a function of the other

$$\boxed{h = \frac{54}{x^2}}$$

because cost of material for top & bottom costs twice as much

cost of top & bottom

cost of 4 sides

objective function $f(x,h) = 2(x^2 + x^2) + 4xh$

Sub. in $h = \frac{54}{x^2} \Rightarrow f(x) = 4x^2 + 4x\left(\frac{54}{x^2}\right) = 4x^2 + \frac{216}{x}$

set $f'(x) = 0 \Rightarrow 8x - \frac{216}{x^2} = 0 \Rightarrow \boxed{x = 3m}$

✓ $f''(x) = 8 + \frac{432}{x^3} > 0$ for all $x > 0$ & $\boxed{h = \frac{54}{3^2} = 6 \ m}$

$\Rightarrow x = 3m$ is a local min. of $f(x)$ ✓

8) $f(x)=x^2e^{-x} \Rightarrow f'(x)=e^{-x}(2x-x^2)=e^{-x}\cancel{x}(2-x)$

$\& f''(x)=e^{-x}(-2x+\cancel{x}^2+2-2x)$

$\Rightarrow f''(x)=e^{-x}(x^2-4x+2)$

a) $L(x)$ @ $a=1$

recall $L(x)=f'(a)(x-a)+f(a)$

$\&$ here, $f(1)=e^{-1}$ $\&$ $f'(1)=e^{-1}=\frac{1}{e}$

$\Rightarrow L(x)=\frac{1}{e}(x-1)+\frac{1}{e} \Rightarrow \boxed{\frac{x}{e}=L(x)}$

simplify

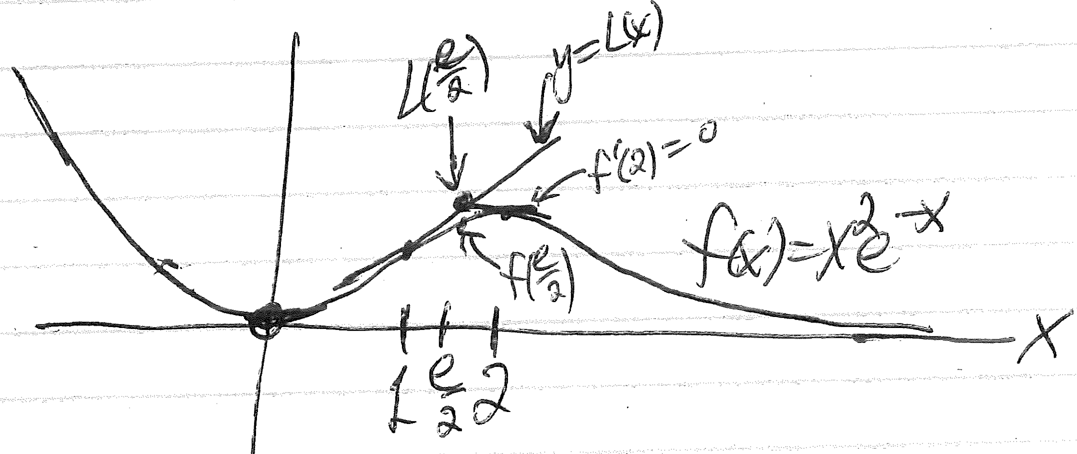
b) $\boxed{L\left(\frac{e}{2}\right)=\frac{1}{2}}$ is the lin. approx. of $f\left(\frac{e}{2}\right)$.

c) @ $a=1$, $f''(1)=\frac{1}{e}(1-4+2)<0$

$=-\frac{1}{e}$ ✓

so, $f(x)$ is conc. down @ $a=1$

$\Rightarrow L(x)$ is an overestimate of $f(x)$ @ $x=\frac{e}{2}$.



$\frac{0}{0}$ 9) a) $\lim\limits_{x\to 1} \dfrac{\ln x}{17x - x^2 - 16} \overset{LHO}{=} \lim\limits_{x\to 1} \dfrac{1/x}{17-2x} = \boxed{\dfrac{1}{15}}$

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9) b) $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{\sqrt{8-x^2}-1}$ $\quad x^2+x-6 \longrightarrow 0$ as $x\to 2$, $\quad \sqrt{8-x^2}-1 \longrightarrow 1$ as $x\to 2$

So, this limit is zero. (one cannot use L'HOP.)

9) c) $\lim\limits_{x\to 0^+} (1+4x)^{\frac{3}{x}}$ $\qquad \left(a^b = e^{\ln(a^b)} = e^{b\ln a}\right)$

first find $\lim\limits_{x\to 0^+} \ln(1+4x)^{\frac{3}{x}}$ & then exponentiate answer

$\infty \cdot 0$

$$\lim_{x\to 0^+} \ln(1+4x)^{3/x} = \lim_{x\to 0^+} \left(\frac{3}{x}\right)\cdot \ln(1+4x)$$

$$= \lim_{x\to 0^+} \frac{3\ln(1+4x)}{x} \qquad \frac{0}{0}$$

$$\overset{LHO.}{=} \lim_{x\to 0^+} \frac{\left(\frac{3}{1+4x}\right)\cdot 4}{1} = \boxed{12}$$

So, final answer is

$\boxed{e^{12}}$ ✓

10) a) $f(x) = \sec^2(x)$

$\Rightarrow \boxed{F(x) = \tan(x) + C}$ $\quad \checkmark\ F' = \sec^2 x = f \checkmark$

10) b) $g(x) = \dfrac{1}{\sqrt{x}} \left(= x^{-1/2}\right)$

$\Rightarrow \boxed{G(x) = 2x^{1/2} + C = 2\sqrt{x} + C}$

$\checkmark\ G' = 2 \cdot \dfrac{1}{2\sqrt{x}} + 0 = \dfrac{1}{\sqrt{x}} = g \checkmark$