SOLUTIONS SECOND PRACTICE TEST (Fall '22) A/H

for Fall 2024

2) a) $\frac{dy}{dx} = 4x^3 e^{-3x} - 3x^4 e^{-3x}$ (product rule & chain rule)

2) b) $\frac{dy}{dx} = \frac{-5\sin(5x)(x^2+4) - 2x\cos(5x)}{(x^2+4)^2}$ (quotient & chain rules)

2) c) $\frac{dy}{dx} = \frac{1}{\sqrt{1-(\sqrt{3x})^2}} \cdot \frac{\sqrt{3}}{2\sqrt{x}}$ (chain rule)

2) d) $\frac{dy}{dx} = \frac{1}{e^{\tan 7x}} \cdot 7\sec^2(7x)\, e^{\tan 7x}$ $(= 7\sec^2 7x)$

(Note: $\ln(e^{\tan 7x}) = \tan(7x)$, so ans. makes sense!!)

4) $\int \frac{3+\sqrt{x}}{x}\,dx = \int \frac{3}{x}\,dx + \int \frac{\sqrt{x}}{x}\,dx$

$= 3\ln|x| + C + \int \frac{1}{\sqrt{x}}\,dx$

$= \boxed{3\ln|x| + 2\sqrt{x} + C}$

(Similar to one of our problems on Midterm 2)

SECOND PRACTICE TEST
(FALL 2022)

B/H

5) $3x^2-2y^2=6-2xy$ $\checkmark$ (2,3) is on the curve

implicit differentiation $\Rightarrow 6x-4y\frac{dy}{dx}=-2y-2x\frac{dy}{dx}$

$\Rightarrow (-4y+2x)\frac{dy}{dx}=-2y-6x$

$\Rightarrow \frac{dy}{dx}=\frac{-2y-6x}{(2x-4y)}$ = slope of tan. line @ general pt (x,y) on curve.

So, @ (2,3), $\frac{dy}{dx}=\frac{-6-12}{(4-12)}=\frac{-18}{-8}=\boxed{\frac{9}{4}}$

6) a) $\lim_{x\to 0}\frac{x-\sin x}{5x^3}$ $\frac{0}{0}$ form.

$\stackrel{LHO}{=}\lim_{x\to 0}\frac{1-\cos x}{15x^2}$ $\frac{0}{0}$ form

$\stackrel{LHO}{=}\lim_{x\to 0}\frac{+\sin x}{30x}$ $\frac{0}{0}$ form

$\stackrel{LHO}{=}\lim_{x\to 0}\frac{\cos x}{30}=\boxed{\frac{1}{30}}$

6) b) $\lim_{x\to\infty}\tan^{-1}(1+e^{-4x})=\boxed{\frac{\pi}{4}}$ b/c $e^{-4x}\to 0$ & $\tan^{-1}(1)=\frac{\pi}{4}$ & $\tan^{-1}$ is continuous

(& recall by Ch. 2.6 that one may push a limit inside a continuous function $\lim_{x\to\infty}f(g(x))=f(\lim_{x\to a}g(x))$ for f continuous

SOLUTIONS

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6) c) $\lim_{x \to 0} \frac{e^{4x}}{5x}$ $\boxed{\text{DNE}}$

because $\lim_{x \to 0^+} \frac{e^{4x}}{5x} = +\infty$

$\lim_{x \to 0^-} \frac{e^{4x}}{5x} = -\infty$

So, the two one-sided limits are different.

8) a) $\int_0^{\pi/9} \cos(3x)\,dx = \frac{1}{3}\sin(3x)\Big|_0^{\pi/9}$ FTC

$= \frac{1}{3}\left(\sin\left(\frac{\pi}{3}\right) - \sin(0)\right)$

$= \boxed{\frac{\sqrt{3}}{6}}$ since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

8) b) $\int_0^1 \frac{x}{1+4x^2}\,dx = \boxed{\frac{\ln 5}{8}}$ u-sub. (so, not on our final)

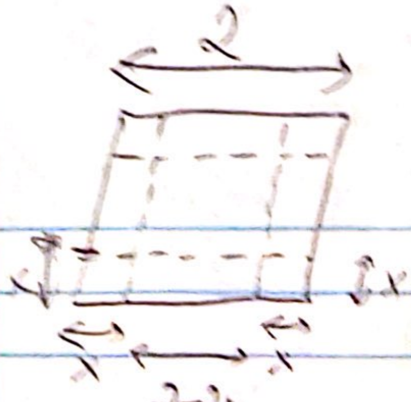
Work

let $u = 1+4x^2 \Rightarrow du = 8x\,dx \Rightarrow x\,dx = \frac{1}{8}du$

(useful for MA104)

$\Rightarrow \int_0^1 \frac{x}{1+4x^2}\,dx = \int_{u=1}^{u=5} \frac{1\,du}{8(u)} = \frac{1}{8}\ln|u|\Big|_1^5$

$= \boxed{\frac{\ln 5}{8}}$ ✓



SOLUTIONS

D/H

9) base of box will be $(2-2x)$ feet by $(2-2x)$ feet & height will be x feet

a) little squares x by x will be cut out from the 4 corners

b) volume of box $V = \underbrace{(2-2x)^2}_{\text{area of base}} \cdot \underbrace{x}_{\text{height}}$

$$\boxed{V(x) = 4x^3 - 8x^2 + 4x}$$ objective function, which we will maximize

c) set $\frac{dV}{dx} = 0 \Rightarrow 12x^2 - 16x + 4 = 0$

$$\Rightarrow 4(3x^2 - 4x + 1) = 0$$

quad formula

$$\Rightarrow x_\pm = \frac{4}{6} \pm \frac{1}{6}\sqrt{16-12}$$

$$\Rightarrow x_\pm = \frac{2}{3} \pm \frac{2}{6} \Rightarrow \begin{cases} x_+ = 1 \\ x_- = 1/3 \end{cases}$$

$\leftarrow$ min vol. b/c cutting away 1 foot on each side leaves a zero by zero box

here, the answer is $x = 1/3$ feet

d) $x = \frac{1}{3}$ is a local max of V b/c $V''\left(\frac{1}{3}\right) = 24\left(\frac{1}{3}\right) - 16 = -8 < 0$

$V'' < 0$. (or use first deriv test)

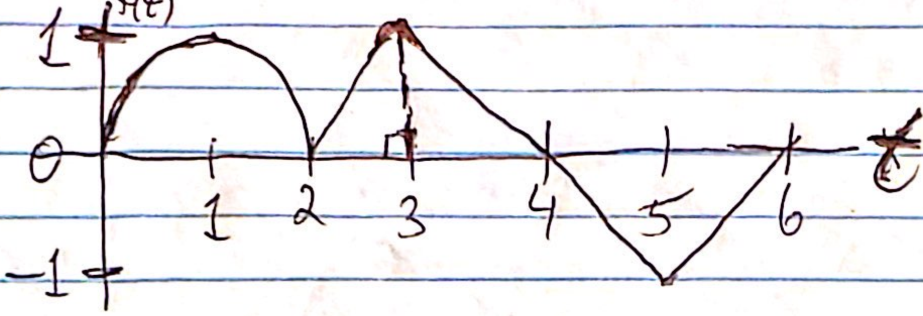
$1/3$

SECOND PRACTICE TEST (Fall 2022)

$\boxed{E/H}$

11)

$G(x)=\int_0^x f(t)\,dt$ & $H(x)=\int_3^x f(t)\,dt$



$G(3)=\int_0^3 f(t)\,dt=\frac{\pi}{2}+\frac{1}{2}$ (= area of $\frac{1}{2}$ circle + area of triangle)

$H(3)=\int_3^3 f(t)\,dt=0$ (b/c interval of integration is [3,3] & hence has zero width.)

$\Rightarrow \boxed{G(3)-H(3)=\frac{\pi+1}{2}}$

Next, $G(6)=\int_0^6 f(t)\,dt=\underbrace{\frac{\pi}{2}}_{\text{from }[0,2]}+\underbrace{\frac{1}{2}}_{[2,3]}+\underbrace{\frac{1}{2}}_{[3,4]}-\underbrace{1}_{[4,6]}=\boxed{\frac{\pi}{2}}$

$H(6)=\int_3^6 f(t)\,dt=\underbrace{\frac{1}{2}}_{[3,4]}-\underbrace{1}_{[4,6]}=\boxed{-\frac{1}{2}}$

hence, $\boxed{G(6)-H(6)=\frac{\pi+1}{2}}$

The fact that both $G(3)-H(3)$ & $G(6)-H(6)$ are the same makes sense because $G(x)-H(x)=\int_0^x f(t)\,dt-\int_3^x f(t)\,dt$
$=\int_0^x+\int_x^3 f(t)\,dt=\int_0^3 f(t)\,dt=\frac{\pi+1}{2}$ for all x !!!!

SOLUTIONS SECOND PRACTICE TEST

F/H

12) a) $\boxed{\int_a^b f'(x)\,dx = f(b) - f(a)}$

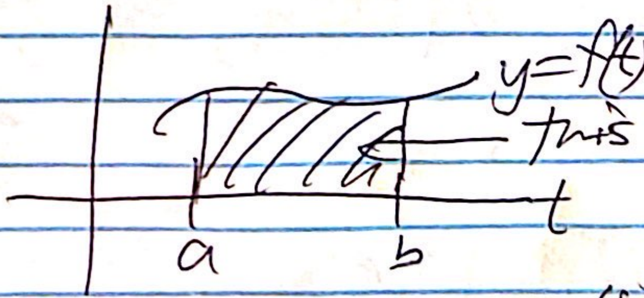
since $f(x)$ is an antiderivative (indef. $\int^{al}$) of $f'(x)$

12) b) $\frac{d}{dx}\left[\int_x^a f(t)\,dt\right] = \frac{d}{dx}\left[-\int_a^x f(t)\,dt\right]$

$= (-1)\frac{d}{dx}\int_a^x f(t)\,dt$

$= \boxed{(-1)f(x)}$ by FTC.

12) c) $\frac{d}{dx}\left[\int_a^b f(t)\,dt\right] = \frac{d}{dx}[\text{constant}] = \boxed{0}$



this area is a constant, since a & b are fixed

(the area is indep. of x)

So, its rate of change is zero as x changes.

G | H

13) $N(x)=\int_0^x f(t)\,dt$ for $0 \le x \le 14$

(net area, also called $A(x)=\int_0^x f(t)\,dt$ in textbook)

(a) $N(0)=0$

(b) $N(1)=-1/2$

FTC $\Rightarrow N'(x)=f(x)$

(4.3) $\Rightarrow N''(x)=f'(x)$

(c) $N'(10)=f(10)$ by FTC
$=6.$ from graph

(d) $N'(8)=f(8)=\underset{(\neq 0)}{3} \Rightarrow$ $\boxed{\text{False}}$

(e) $N''(10)=f'(10)=0 \Rightarrow$ TRUE b/c $f'(x)>0$ to left of 10 & $f'(x)<0$ to right of 10.

(f) $\boxed{F}$, $N'(x)=f(x)<0$ on $(0,1) \Rightarrow N(x)$ ↓ing on $(0,1)$

(g) $\boxed{T}$ b/c $N'(12)=f(12)=0$
& $N'(x)>0$ for x just left of 12
& $N'(x)=f(x)<0$ for just right of 12

(h) $\boxed{T}$ b/c $N'(x)=f(x)<0$ on $(12,14)$

(i) $\boxed{F}$ $N''(x)=f'(x)>0$ on both sides of $x=8$.

(j) $N''(x)=f'(x)>0$ on $(0,4),(6,8),(8,10),(13,14)$

SOLUTIONS (last page)

$\boxed{\text{M/H}}$

15) Correct ordering is

$$\boxed{\ln(3x) \ll x^{10} \ll e^x \ll 5^x}$$

$$f_2 \ll f_3 \ll f_1 \ll f_4 \checkmark$$

$\lim_{x\to\infty} \frac{\ln(3x)}{x^{10}} = 0$ (use L'HOP's rule once)

$\Rightarrow x^{10}$ grows faster than $\ln(3x)$

$\lim_{x\to\infty} \frac{x^{10}}{e^x} = 0$ (use L'HOP's rule ten times)

$\Rightarrow e^x$ grows faster than x^{10}.

$\lim_{x\to\infty} \frac{e^x}{5^x} = 0$ (calculate as $\lim_{x\to\infty} \left(\frac{e}{5}\right)^x = 0$

& since $\frac{e}{5} < 1$ we know multiplying a # that is less than 1 by itself only often, we get zero

ex $(0.1)(0.1) = (0.01)$,
$(0.1)(0.1)(0.1) = 0.001$, ...

OR observe $\lim_{x\to\infty} \frac{e^x}{5^x} \overset{LHOP}{=} \lim_{x\to\infty} \frac{e^x}{(\ln 5)5^x} = \frac{1}{(\ln 5)} \lim_{x\to\infty} \frac{e^x}{5^x}$

& the only way this can be true is if $\lim_{x\to\infty} \frac{e^x}{5^x} = 0. \checkmark$