MA 123 – Fall 2023
Midterm Examination 1
6:30–8:30 pm Thursday, October 12

1. (3 points)

First Name : SO
Last (Family) Name LUTIONS
BU ID number : $e^{\pi i}+1=0$

Directions

All of your work must be shown in this exam booklet.

Phones (of any type) must be turned OFF and in your backpack (silent mode is not permitted).

Backpacks and bags of any type must be stored in the front of the room. Keep your BU ID with you.

Books, notes, extra papers are not permitted.

The use of any electronic, mechanical, photonic, or quantum device to carry out any calculations or any steps in any part of the solutions is not permitted.

Please do not separate the pages of this exam booklet.

If you have a question about a problem, please raise your hand and a proctor will come to your seat to answer it.

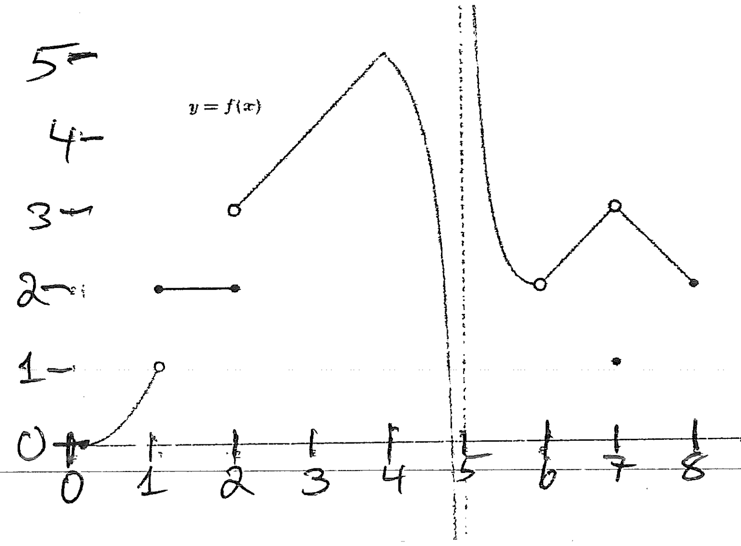
Answers that are written logically and clearly will receive higher scores.

There are nine calculus problems on the exam and one logistical problem that simply asks for your first name, last (family) name, and BU ID number. The logistical problem is problem 1, at the top of this page. The calculus problems are numbered 2–10, starting on the second page of this exam booklet.

The entire exam booklet consists of 10 (ten) pages, double-sided, including this cover page. Please make sure that your exam booklet includes all 5 pieces of paper.

by Solutions

2. **(7 points: 1 for each part)** Here is the graph $y = f(x)$ of a function $f(x)$. Consider all $x \in [0, 8]$.



For each of the following limits, place your answer in the box. If the limit is a number, put the number in the box. If the limit is $+\infty$ or $-\infty$, write that in the box. If the limit does not exist and is not $\pm\infty$, then write DNE in the box.

a. $\lim_{x \to 1^-} f(x) = \boxed{1}$

b. $\lim_{x \to 1^+} f(x) = \boxed{2}$

c. $\lim_{x \to 6} f(x) = \boxed{2}$

d. $\lim_{x \to 7} f(x) = \boxed{3}$

e. $\lim_{x \to 3} f(x) = \boxed{4}$

f. $\lim_{x \to 5^+} f(x) = \boxed{+\infty}$

g. $\lim_{x \to 2} f(x) = \boxed{\text{DNE}}$

by Solutions

3. (12 points: 3 for each part) In this problem, you are to consider the function

$$s(t) = -16t^2 + 96t$$ ← This is the same function studied in Ch. 2.1 & in Ch. 3.

which gives the height (measured in feet) of a rock, thrown vertically straight up from the ground at time $t = 0$. Time ($t \geq 0$) is measured in seconds. For each of the following four parts, you must include **UNITS** in your answer and show enough work to justify your answer. You may use any method that is appropriate and that has been learned in class so far from Chapters 2.1–2.6 and 3.1–3.7.

a. Find the average velocity of the rock on the time interval [1, 2].

$V_{AV} = \frac{s(b)-s(a)}{b-a}$ on $[a,b]$

$$V_{AV} = \frac{s(2)-s(1)}{2-1} = \frac{128-80}{1} = \boxed{48\ \tfrac{ft}{sec}}$$

$$\left(\begin{aligned} s(2) &= -16(2)^2 + 96(2) = -64+192 = 128 \\ s(1) &= -16(1)^2 + 96(1) = 80 \end{aligned}\right)$$

b. Find the instantaneous velocity of the rock at time $t = 2$.

$$V_{INSTAN} = \lim_{b \to a} \frac{s(b)-s(a)}{b-a} = \lim_{h \to 0} \frac{s(a+h)-s(a)}{h} = s'(a)$$

Ch. 2.1
Ch. 3.1
& Ch. 3.6

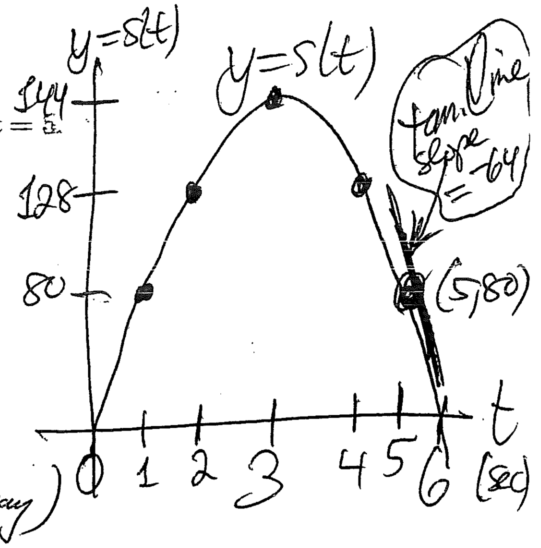
Here, $s'(t) = -32t + 96 \Rightarrow \boxed{s'(2) = 32\ \tfrac{ft}{sec} = V_{INSTAN}}$

c. Find the time at which the velocity of the rock is equal to zero.

$$V(t) = s'(t) = -32t + 96$$

So, $\boxed{V(t) = 0 \text{ @ } t = 3 \text{ sec}}$

Graph is useful, but not required



d. Find the slope of the tangent line to the graph of $y = s(t)$ at $t = 5$.

$$m_{tan} = s'(5) = -160 + 96 = \boxed{-64\ \tfrac{ft}{sec}}$$

(5,80) (I-580 is a great interstate highway)

by Solutions

4. (16 points: 4 for each part)
For each of the following four limits, if it exists, evaluate it using only techniques learned so far in class this semester. If the limit is infinite, indicate if it is $+\infty$ or $-\infty$. If the limit does not exist and is NOT infinite, write DNE. Show enough work to justify your answer.

a. $\displaystyle\lim_{x\to 2}\frac{x^2+x-6}{x^2+2x-8} = \lim_{x\to 2}\frac{(x+3)\cancel{(x-2)}}{(x+4)\cancel{(x-2)}} = \lim_{x\to 2}\frac{x+3}{x+4} = \boxed{\frac{5}{6}}$

factor

OK to cancel (note $x\neq 2$)

quotient limit law

b. $\displaystyle\lim_{x\to\frac{3}{4}^+}\frac{-x}{\sqrt{4x-3}}$

here, $-x \to -\frac{3}{4}$ as $x \to \frac{3}{4}^+$

& $\sqrt{4x-3} \to 0^+$ as $x\to\frac{3}{4}^+$ & $\frac{\text{neg}}{\text{pos}} = \text{neg}.$

So, $\displaystyle\lim_{x\to\frac{3}{4}^+}\frac{-x}{\sqrt{4x-3}} = \boxed{-\infty}$

c. $\displaystyle\lim_{x\to+\infty}\frac{7x^3-x^2+4}{5x^3+5x+1} = \boxed{\frac{7}{5}}$

May observe: highest powers on top & bottom are the same so, the ratio $\to$ ratio of coefficient

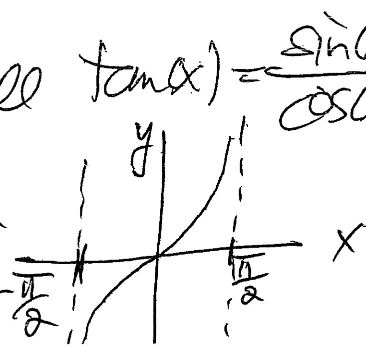
or, calculate: $\displaystyle\lim_{x\to\infty}\frac{7x^3-x^2+4}{5x^3+5x+1}\cdot\frac{\left(\frac{1}{x^3}\right)}{\left(\frac{1}{x^3}\right)} = \lim_{x\to\infty}\frac{7-\frac{1}{x}+\frac{4}{x^3}}{5+\frac{5}{x^2}+\frac{1}{x^3}} = \frac{7}{5}$

$\frac{1}{x}\to 0$, $\frac{4}{x^3}\to 0$, $\frac{5}{x^2}\to 0$, $\frac{1}{x^3}\to 0$ as $x\to\infty$.

d. $\displaystyle\lim_{x\to\frac{\pi}{2}^-}\tan(x) = \boxed{+\infty}$

recall $\tan(x) = \frac{\sin(x)}{\cos(x)}$ & $\begin{cases}\displaystyle\lim_{x\to\frac{\pi}{2}^-}\sin(x) = 1\\ \displaystyle\lim_{x\to\frac{\pi}{2}^-}\cos(x) = 0^+\end{cases}$

or, graph $y=\tan(x)$



by SoLutions

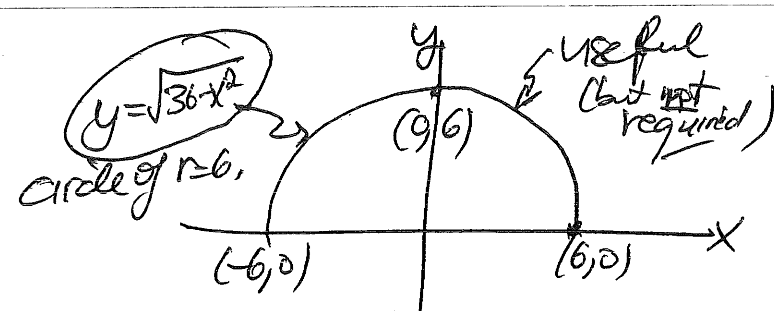
5. (8 points: 4 for each part)

a. Write down all of the INTERVALS on which the following function $f(x)$ is continuous:
$f(x) = \sqrt{36 - x^2}$

$\boxed{f(x) \text{ is continuous on } [-6,6]}$

Recall $\sqrt{\ }$ is only defined for zero & positive #s (NOT for negative #s)

So, $f(x)$ is NOT defined (& also not continuous) for $x < -6$ & for $x > 6$



b. Write down all of the POINTS at which the following function $g(x)$ has discontinuities:

$$g(x) = \frac{3x-1}{3x^2-13x+4} = \frac{3x-1}{(3x-1)(x-4)}$$

← g(x) factors & is a rational function (ratio of two polynomials)

⇒ denom. zero @ $x = \frac{1}{3}$ & @ $x = 4$

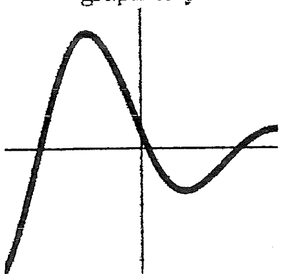
⇒ $\boxed{g(x) \text{ is discontinuous @ } x = \frac{1}{3} \text{ \& @ } x = 4}$

These are the only two points of discontinuity, because the denom. is non-zero for all other values of x, and we know rational functions are continuous whenever the denom. is NOT zero. (Ch. 2.6)

by Solutions

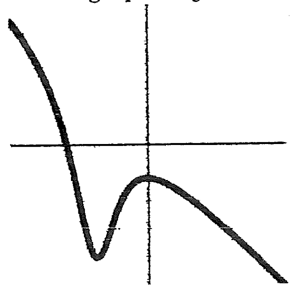
6. (9 points: 3 for each part) For each function that is graphed in parts (a)-(c), write in the box below it which graph (1-9) is the graph of its derivative.

(a.) graph of f



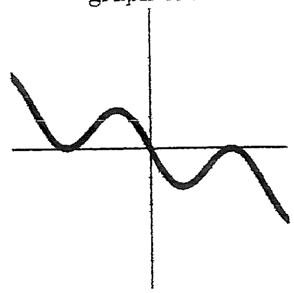
$\boxed{4}$

(b.) graph of g



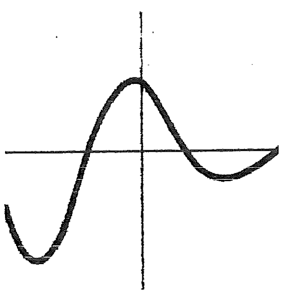
$\boxed{3}$

(c.) graph of h

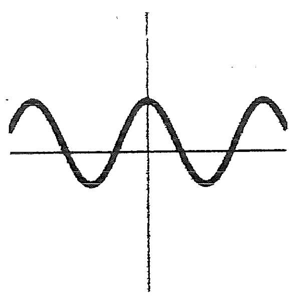


$\boxed{7}$

(1)



(2)



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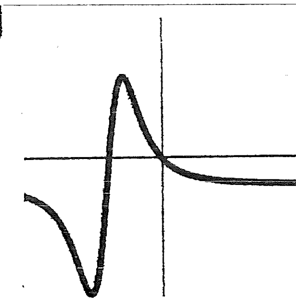
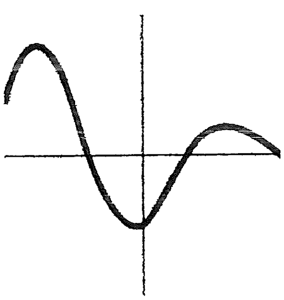
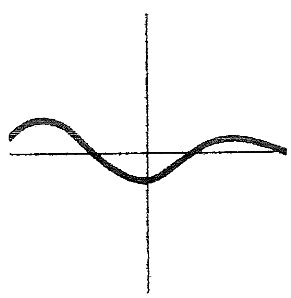


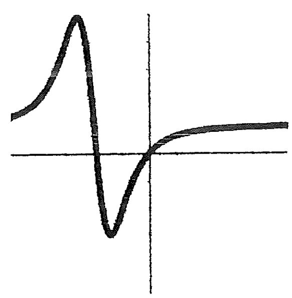
(4)



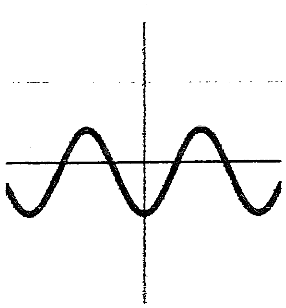
(5)



(6)



(7)



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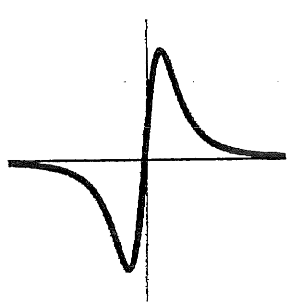
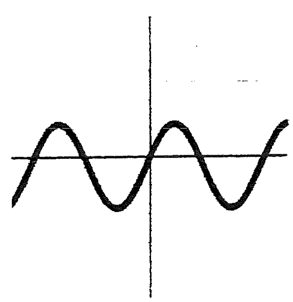


(9)



by Solutions

7. (12 points) In this problem, you are to consider the function

$$f(x) = \frac{x}{x+1}$$

Using only the definition of the derivative and the limit theorems learned so far this semester in the class, calculate $f'(2)$. You will not receive any credit if you do not start with the limit definition of $f'(x)$ or if you use limit theorems not studied so far this semester in class.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$ general definition of derivative

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{\left(\frac{x+h}{x+h+1}\right) - \left(\frac{x}{x+1}\right)}{h}$$ ← we used $f(x) = \frac{x}{x+1}$ in the definition

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{\left[\frac{(x+h)(x+1) - (x)(x+h+1)}{(x+h+1)(x+1)}\right]}{h}$$ ← we put both terms in the numerator over a common denominator

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{x^2+hx+x+h-x^2-xh-x}{h(x+h+1)(x+1)}$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{h}{h(x+h+1)(x+1)}$$ ← simplified the numerator

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{1}{(x+h+1)(x+1)}$$ ← cancelled the factors of h top & bottom (ok, since $h \neq 0$)

$$\Rightarrow \boxed{f'(x) = \frac{1}{(x+1)^2}}$$

$$\Rightarrow \boxed{f'(2) = \frac{1}{(2+1)^2} = \frac{1}{9}}$$

Note: Ok to sub. in $x=2$ from the beginning, but must also have all the steps correct.

by Solutions

8. (12 points: 3 for each part) In each part, calculate the derivative $\frac{dy}{dx}$ by any appropriate method that we have learned so far. Please do NOT simplify your answers.

a. $y = \frac{e^{-x^2}}{x^3+5}$ quotient rule, chain rule (for numerator)

$$f = e^{-x^2}$$
$$g = x^3+5$$
$$f' = -2xe^{-x^2}$$
$$g' = 3x^2$$

$$\frac{dy}{dx} = \frac{gf' - fg'}{g^2}$$

$$\boxed{\frac{dy}{dx} = \frac{-2xe^{-x^2}(x^3+5) - e^{-x^2}(3x^2)}{(x^3+5)^2}}$$

b. $y = \pi^5$ constant rule $(c = \pi^5)$

$$\boxed{\frac{dy}{dx} = \frac{d}{dx}(\pi^5) = 0}$$

This problem is straight from the review worksheet.

c. $y = \tan\left(e^{\frac{1}{x}}\right)$ chain rule $y = f(g(x)) \Rightarrow y' = f'(g(x))g'(x)$

& here $f(u) = \tan(u)$, $u = g(x) = e^{1/x}$

$\Rightarrow f'(u) = \sec^2(u)$, $g'(x) = -\frac{1}{x^2}e^{1/x}$

$$\Rightarrow \boxed{\frac{dy}{dx} = \left(\sec^2(e^{1/x})\right)\left(-\frac{1}{x^2}e^{1/x}\right)}$$

This problem is also straight from the review worksheet.

d. $y = \cos^3\left(x^2 \sin(x)\right)$ chain rule & product rule

$$\boxed{\frac{dy}{dx} = \left(3\cos^2(x^2\sin(x))\right)\left(-\sin(x^2\sin(x))\right)\left(2x\sin(x) + x^2\cos(x)\right)}$$

by So Lutions

9. (9 points)

In this problem, you are working with the function $f(x) = \dfrac{2x^2}{x+1}$. Calculate an equation of the tangent line to the graph of $y = f(x)$ at $x = 1$. Write your answer either in slope-intercept form or in point-slope form, and show all of your work.

$y = mx + b$ (slope intercept form)

$m = \dfrac{y_2 - y_1}{x_2 - x_1}$ (pt.-slope form)

$\boxed{m = f'(x=1)}$ since it's the slope of the tan. line

& here, $f'(x) = \dfrac{4x(x+1) - 2x^2(1)}{(x+1)^2} = \dfrac{2x^2+4x}{(x+1)^2}$

$\Rightarrow \boxed{f'(1) = \frac{6}{4} = \frac{3}{2}} = m$

also, the tangent line goes through the point $(1, f(1)) = (1, 1)$ (b/c $f(1) = \dfrac{2\cdot(1^2)}{(1+1)} = 1$ ✓)

hence, $\boxed{y = \frac{3}{2}x - \frac{1}{2}}$ slope-intercept form of eq. for tan. line

& $\boxed{\dfrac{3}{2} = \dfrac{y-1}{x-1}}$ pt-slope form of eq. for tan. line

(same as above slope-intercept form b/c $\dfrac{3}{2} = \dfrac{y-1}{x-1} \Rightarrow \dfrac{3}{2}(x-1) = y-1$
$\Rightarrow y = \frac{3}{2}x - \frac{1}{2}$ ✓)

by SoLutions

10. (12 points: 3 for each part) In this problem, you are given the following table of data about two functions $f(x)$ and $g(x)$ and their derivatives $f'(x)$ and $g'(x)$ at the values of x indicated across the top row of the table. In each part, you must state the value of the indicated derivative.

	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$
$f(x)$	0	3	5	1	0
$f'(x)$	5	2	-5	-8	-10
$g(x)$	4	5	1	3	2
$g'(x)$	2	10	20	15	20

a. $(fg)'(2)$ product rule $(fg)'=f'g+fg'$

So, $(fg)'(2)=f'(2)g(2)+f(2)g'(2)=2\cdot5+3\cdot10=\boxed{40}$

b. $\left(\frac{f}{g}\right)'(3)$ quotient rule $\left(\frac{f}{g}\right)'=\frac{gf'-fg'}{g^2}$

So, $\left(\frac{f}{g}\right)'(3)=\frac{g(3)f'(3)-f(3)g'(3)}{(g(3))^2}=\frac{(1)(-5)-(5)(20)}{(1)^2}=\boxed{-105}$

c. $(f(g(x)))'(2)$ chain rule $(f(g(x)))'=f'(g(x))g'(x)$

So, $(f(g(x)))'(2)=f'(g(2))\cdot g'(2)=f'(5)\cdot10=(-10)\cdot(10)=\boxed{-100}$

d. $(g(f(x)))'(4)$ chain rule $(g(f(x)))'=g'(f(x))f'(x)$

(deriv. of outer) $\cdot$ (deriv. of inner)

So, $(g(f(x)))'(4)=g'(f(4))\cdot f'(4)=g'(1)\cdot(-8)=(2)\cdot(-8)=\boxed{-16}$