Grönwall's Inequality Homework

1. **2015 Midterm question:** Let $\varphi_1 : [0,T] \to [0,\infty)$, $\varphi_2 : [0,T] \to [0,\infty)$. Suppose $c(t)$ is continuously differentiable (i.e., c is continuous and $\frac{dc}{dt}$ is continuous) and that $\frac{dc}{dt} \geq 0$ for all $t \in [0,T]$. Suppose also that φ_1 satisfies the inequality

$$\varphi_1(t) \leq c(t) + \int_0^t \varphi_1(s)\,\varphi_2(s)\,ds, \quad \forall\, t \in [0,T]. \tag{1}$$

You will prove that if φ_1 satisfies (1), then

$$\varphi_1(t) \leq c(t) \exp\left\{\int_0^t \varphi_2(s)\,ds\right\}, \quad \forall\, t \in [0,T], \tag{2}$$

where $\exp$ denotes the exponential function, as follows.

(i). Define $G(t) := c(t) + \int_0^t \varphi_1(s)\,\varphi_2(s)\,ds$. Write down (1) in terms of φ_1 and G.

(ii). Differentiate $G(t)$ with respect to t. Show that G satisfies the differential inequality

$$\frac{d}{dt}\left(G(t)e^{-\int_0^t \varphi_2(s)\,ds}\right) \leq \frac{dc}{dt}e^{-\int_0^t \varphi_2(s)\,ds}, \quad \forall\, t \in [0,T].$$

(iii). Explain why $0 < e^{-\int_0^t \varphi_2(s)\,ds} \leq 1$ for all $t \in [0,T]$. Hence show that

$$\frac{d}{dt}\left(G(t)e^{-\int_0^t \varphi_2(s)\,ds}\right) \leq \frac{dc}{dt}, \quad \forall\, t \in [0,T].$$

Hint: recall that $\exp$ is monotone increasing and consider the range of φ_2.

(iv). Hence show that φ_1 satisfies (2).
Hint: use the Fundamental Theorem of Calculus. In case you've forgotten, the Fundamental Theorem of Calculus says: if f' is the derivative of f, then

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

2. Suppose $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and that there exists a continuous function $c : [0,\infty) \to [0,\infty)$ such that

$$|f(t,x)| \leq c(t)(1 + |x|)$$

for all $x \in \mathbb{R}$ and $t \geq 0$. Use Grönwall's inequality to show that every solution to

$$\frac{dx}{dt} = f(x,t)$$

is bounded on each finite time interval $I = [0,T]$, for some positive and finite T.

Remark: You just proved that if f grows at most linearly in x, then solutions cannot blow up in finite time.