Eigenvalue-Eigenvector Homework

Note: The first three questions on this homework sheet are crucial to the results we use in lectures.

1. Suppose $A \in \mathbb{R}^{2\times 2}$ has distinct eigenvalues (i.e., $\lambda_1 \neq \lambda_2$). Let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ denote the eigenvectors associated to the eigenvalues λ_1 and λ_2, respectively. Prove that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent. **Hint:** consider the equation

$$\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 = \mathbf{0},$$

where α_1 and α_2 are to be determined. To show linear independence, you need to show that both α_1 and α_2 are zero. Multiply this equation by λ_1 (or λ_2) and then multiply this equation by the matrix A, and then take the difference of the two new equations you derived.

2. Let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ be linearly independent vectors in $\mathbb{R}^2$. Let P be the matrix with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ as columns, i.e., $P = (\boldsymbol{v}_1\ \boldsymbol{v}_2)$. Prove that P is invertible, i.e., prove that

$$\det P \neq 0.$$

3. Let $A \in \mathbb{R}^{2\times 2}$, and let $P \in \mathbb{R}^{2\times 2}$ be an invertible matrix. Prove that A and PAP^{-1} have the same eigenvalues. **Hint:** for this problem, you will need the following useful property: if A and B are square matrices of the same size, then

$$\det(AB) = (\det A)(\det B).$$

4. Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

   (i). Compute the eigenvalue-eigenvector pairs of A.

   (ii). Diagonalize the matrix A.

5. Consider the standard rotation matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

for $0 < \theta < \pi$. Show that R has no real eigenvalues. Can you explain this geometrically?