Non-dimensionalization & Codimension-1 Bifurcations Homework

1. For each of the following, $\mu \in \mathbb{R}$ is a parameter. In each case, sketch the bifurcation diagram.

   (i). $\dot{x} = \mu + x^2$.
   (ii). $\dot{x} = \mu x - x^2$.
   (iii). $\dot{x} = \mu x - x^3$.
   (iv). $\dot{x} = \mu x + x^3$.

   Locate and classify all bifurcations.

2. The fox squirrel is a small mammal native to the Rocky Mountains. These squirrels are very territorial, so if their population is large, their rate of growth decreases and may even become negative. On the other hand, if the population is too small, fertile adults run the risk of not being able to find suitable mates, so again the rate of growth is negative. These assumptions are formulated as the following model:

$$\frac{dP}{dt} = kP\left(\frac{P}{M} - 1\right)\left(1 - \frac{P}{K}\right),$$

   where $k > 0, 0 < M \ll K$, and r, M, K are all constants.

   (i). Show that this can be written in dimensionless form as

$$\frac{dx}{d\tau} = x(mx - 1)(1 - x).$$

   What is m in terms of r, M, and K? What can you say about the magnitude of m?
   (ii). Using (i), sketch the phase portrait and determine all equilibria and their stability.
   (iii). Using (i), find and determine the equilibria by linearization. Hence sketch the phase line.

3. Consider the following harvesting model

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - EP,$$

   where the population P is harvested at constant effort E at a rate EP.

   (i). Transform to dimensionless form such that the dynamics depend on a single dimensionless parameter $\mu := E/r$.
   (ii). Perform a linearized stability analysis of all equilibria.
   (iii). Sketch the bifurcation diagram. Are there bifurcations in this model? If yes, indicate the locations of these bifurcation points on your diagram.
   (iv). Interpret your results with respect to the original variables and parameters.

4. A common type of equation that arises in neuroscience is the following

$$\frac{dv}{dt} = v - \frac{v^3}{3} - w,$$

   where v represents a (dimensionless) voltage (across the membrane of a cell) and w is a (dimensionless) parameter. (This is a special case of the FitzHugh-Nagumo model for excitation.)

   (i). Locate and classify all bifurcations.
   (ii). Sketch the bifurcation diagram.
   (iii). Locate and classify all bifurcation points on your diagram.