Matrix Trig Functions and Simple Harmonic Motion Homework

For a real number t, we can express the trig functions $\sin(t)$ and $\cos(t)$ in terms of complex exponentials:

$$\sin(t) = \frac{e^{it} - e^{-it}}{2i}, \qquad \text{and} \qquad \cos(t) = \frac{e^{it} + e^{-it}}{2}.$$

Motivated by this, we define the sine and cosine of a matrix, $A \in \mathbb{R}^{2\times 2}$, using the matrix exponential:

$$\sin(A) := \frac{\exp(iA) - \exp(-iA)}{2i}, \quad \text{and} \quad \cos(A) := \frac{\exp(iA) + \exp(-iA)}{2}.$$

(i) Show that $\sin^2(A) + \cos^2(A) = I$, where I is the 2×2 identity matrix. You may assume that

$$\exp(-A) = (\exp(A))^{-1}.$$

(ii) Show that $\frac{d}{dt}\sin(At) = A\cos(At)$, and that $\frac{d}{dt}\cos(At) = -A\sin(At)$.
(You will first need to show that $\frac{d}{dt}\exp(At) = A\exp(At)$.)

Simple harmonic motion occurs when a restoring force is applied to a system that has been displaced from equilibrium, and the strength of the restoring force is proportional to the displacement (but acts in the opposite direction to the displacement). Examples include: a mass attached to the end of a spring, and the (idealized) motion of tides. The ODE that describes simple harmonic motion is

$$\ddot{x} + a^2 x = 0, \tag{1}$$

where $x \in \mathbb{R}$ is the displacement, $a > 0$ is a constant, and $-a^2 x$ is the restoring force.
Remark: does equation (1) look familiar? (It's essentially Newton's second law.)

(iii) Show that the general solution of (1) is

$$x(t) = \frac{\sqrt{H}}{a}\sin(at + C),$$

where $H > 0$ and C are constants. **Optional:** if you want a challenge, rewrite (1) as a first-order system (by introducing the velocity $y = \dot{x}$) and prove (by integration) that the general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}\cos(at) + \frac{1}{a}\begin{pmatrix} C_2 \\ -aC_1 \end{pmatrix}\sin(at),$$

where C_1 and C_2 are arbitrary constants.

We now generalize (1) to the 2-dimensional case. Suppose that $A \in \mathbb{R}^{2\times 2}$ is invertible.

(iv) Show that the general solution of

$$\ddot{\boldsymbol{x}} + A^2\boldsymbol{x} = \boldsymbol{0}, \tag{2}$$

is given by

$$\boldsymbol{x}(t) = \cos(At)\boldsymbol{x}_0 + A^{-1}\sin(At)\boldsymbol{v}_0,$$

where $\boldsymbol{x}_0$ and $\boldsymbol{v}_0$ are vectors of initial displacements and velocities, respectively.
Remark: if A is a diagonal matrix, then (2) describes two simple harmonic oscillators moving independently of each other. If A is not a diagonal matrix, then (2) describes the motion of (linearly) coupled oscillators (e.g., two masses wedged between three springs).

(v) **Optional:** a natural question to ask is: how do we actually compute $\sin(At)$ and $\cos(At)$? Assume that A has distinct eigenvalues, i.e., $\lambda_1 \neq \lambda_2$. Let P be the matrix of eigenvectors of A. Show that

$$\sin(At) = P\begin{pmatrix} \sin(\lambda_1 t) & 0 \\ 0 & \sin(\lambda_2 t) \end{pmatrix}P^{-1}, \quad \text{and} \quad \cos(At) = P\begin{pmatrix} \cos(\lambda_1 t) & 0 \\ 0 & \cos(\lambda_2 t) \end{pmatrix}P^{-1}.$$