Nullclines Homework

1. Consider the nonlinear system

$$\begin{aligned} \dot{x} &= -2xy, \\ \dot{y} &= (1+x)(1-y). \end{aligned}$$

   (i). Sketch the nullclines and find the equilibria.
   (ii). Sketch the vector field along the nullclines.
   (iii). For each equilibrium, write down the linear approximation of the system.
   (iv). Try to sketch the phase plane of this system.

2. Consider the nonlinear system

$$\begin{aligned} \dot{x} &= y^2 - x^2, \\ \dot{y} &= x - 1. \end{aligned}$$

   (i). Sketch the nullclines and find the equilibria.
   (ii). Sketch the vector field along the nullclines.
   (iii). For each equilibrium, write down the linear approximation of the system.
   (iv). Try to sketch the phase plane of this system.

3. Consider the nonlinear system

$$\begin{aligned} \dot{x} &= x^2 + y^2 - 1, \\ \dot{y} &= x + y. \end{aligned}$$

   (i). Sketch the nullclines and find the equilibria.
   (ii). Sketch the vector field along the nullclines.
   (iii). For each equilibrium, write down the linear approximation of the system.
   (iv). Try to sketch the phase plane of this system.

4. Consider the nonlinear system

$$\begin{aligned} \dot{x} &= x, \\ \dot{y} &= -y + x^2. \end{aligned}$$

   (i). Linearise about the equilibrium at the origin and hence sketch the phase plane.
   (ii). Show that the solution of the system is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^t x_0 \\ e^{-t} y_0 + \frac{1}{3}\left(e^{2t} - e^{-t}\right) x_0^2 \end{pmatrix}.$$

   (iii). The solution you computed in part (ii) is parametrized by t. Eliminate the t variable and show that the solution can be written as

$$y = \frac{H}{x} + \frac{1}{3}x^2,$$

   where $H \in \mathbb{R}$ is some constant.
   (iv). Sketch the curves $y = \frac{H}{x} + \frac{x^2}{3}$ for various values of H.
   Remark: H is called the Hamiltonian of the system, and is common in physics (e.g., general relativity) or whenever a quantity is conserved. Here, the Hamiltonian allows us to exactly sketch the solutions in the phase plane.