Solving PDEs Using Laplace Transforms Homework

1. Find the solution of the advection equation with source
$$x\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = x,$$
for $x > 0, t > 0$, with the initial and boundary conditions
$$\begin{aligned} u(x,0) &= 0, \quad \text{for} \quad x > 0, \\ u(0,t) &= 0, \quad \text{for} \quad t > 0. \end{aligned}$$

2. The conduction of heat in a semi-infinite medium is described by the diffusion equation
$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2},$$
where $\kappa > 0$ is a constant. Solve this equation with the initial and boundary conditions
$$\begin{aligned} u(x,0) &= 0, \qquad \text{for} \quad x > 0, \\ u(0,t) &= f(t), \quad \text{for} \quad t > 0, \\ u(x,t) &\to 0, \qquad \text{as} \quad x \to \infty,\ t > 0, \end{aligned}$$
Leave your answer as a convolution integral.

3. Find the solution of the inhomogeneous wave equation
$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = k\sin\left(\frac{\pi x}{a}\right),$$
for $0 < x < a, t > 0$, with the initial and boundary conditions
$$\begin{aligned} u(x,0) = 0 &= \frac{\partial u}{\partial t}(x,0), \quad \text{for} \quad 0 < x < a, \\ u(0,t) = 0 &= u(a,t), \quad \text{for} \quad t > 0, \end{aligned}$$
where c, k, and a are constants.

4. Consider a transmission line which is a model of co-axial cable containing resistance R, inductance L, capacitance C, and leakage conductance G. The current $I(x,t)$ and potential $V(x,t)$ at a point x and time t in the line satisfy the equations
$$\begin{aligned} L\frac{\partial I}{\partial t} + RI &= -\frac{\partial V}{\partial x}, \\ C\frac{\partial V}{\partial t} + GV &= -\frac{\partial I}{\partial x}. \end{aligned}$$
   (i) **Optional:** eliminate either V or I and show that the remaining variable satisfies the *telegraph equation*
$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial t} + bu = 0, \tag{1}$$
   where u is either I or V, $c^2 = (LC)^{-1}$, $a = LG + RC$, and $b = RG$.

(ii) Solve the telegraph equation (1) for a lossless transmission line ($R = 0$ and $G = 0$) with the initial and boundary conditions

$$
\begin{aligned}
&u(x, 0) = 0 = \frac{\partial u}{\partial t}(x, 0), \quad \text{for } 0 < x < \infty, \\
&u(0, t) = V_0 \cos \omega t, \quad \text{for } t > 0, \\
&u(x, t) \to 0, \quad \text{as } x \to \infty,\ t > 0.
\end{aligned}
$$

(iii) For the Heaviside distortionless cable, $\frac{R}{L} = \frac{G}{C} = k =$ constant. Show that the telegraph equation in this case can be written as

$$
\frac{\partial^2 u}{\partial t^2} + 2k \frac{\partial u}{\partial t} + k^2 u = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{2}
$$

Solve (2) with the initial and boundary data

$$
\begin{aligned}
&u(x, 0) = 0 = \frac{\partial u}{\partial t}(x, 0), \quad \text{for } 0 < x < \infty, \\
&u(0, t) = V_0 f(t), \quad \text{for } t > 0, \\
&u(x, t) \to 0, \quad \text{as } x \to \infty,\ t > 0.
\end{aligned}
$$

Remark: the solution is

$$
V(x, t) = V_0 \exp\left(-\frac{kx}{c}\right) f\left(t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right).
$$

This solution represents a signal that propagates with velocity $c = 1/\sqrt{LC}$ with exponentially decaying amplitude, but with no distortion. Thus, the signal can propagate along the distortionless line over long distances if appropriate boosters are placed at regular intervals in order to increase the strength of the signal as to counteract the effects of attenuation.