Nonlinear Phase Planes Homework

Note: "linear stability analysis" means (i) write down the linearization, and (ii) classify the equilibrium and hence determine its (linear) stability.

In problems 1–4, you will revisit the homework problems from the Handout in Week 06, but this time, you have the full toolbox of methods we have developed at your disposal.

1. Consider the nonlinear system

$$\begin{aligned}\dot{x} &= -2xy,\\ \dot{y} &= (1+x)(1-y).\end{aligned}$$

   (i). Sketch the nullclines and find the equilibria.
   (ii). Perform a linear stability analysis of each equilibrium.
   (iii). Carefully sketch the phase plane.

2. Consider the nonlinear system

$$\begin{aligned}\dot{x} &= y^2 - x^2,\\ \dot{y} &= x - 1.\end{aligned}$$

   (i). Sketch the nullclines and find the equilibria.
   (ii). Perform a linear stability analysis of each equilibrium.
   (iii). Carefully sketch the phase plane.

3. Consider the nonlinear system

$$\begin{aligned}\dot{x} &= x^2 + y^2 - 1,\\ \dot{y} &= x + y.\end{aligned}$$

   (i). Sketch the nullclines and find the equilibria.
   (ii). Perform a linear stability analysis of each equilibrium.
   (iii). Carefully sketch the phase plane.

4. Consider the nonlinear system

$$\begin{aligned}\dot{x} &= x,\\ \dot{y} &= -y + x^2.\end{aligned}$$

   (i). Sketch the nullclines and find the equilibria.
   (ii). Perform a linear stability analysis of each equilibrium.
   (iii). Carefully sketch the phase plane.
   Note: in the Week 06 Homework, you found explicit formulae for the solution curves. Namely, the trajectories are generated by

$$y = \frac{H}{x} + \frac{1}{3}x^2,$$

   where $H \in \mathbb{R}$ is an arbitrary constant. Compare the phase plane you just sketched with these analytic solution curves. Which was easier to produce?

Problems 5 and 6 give additional practice with constructing nonlinear phase planes.

5. Consider the nonlinear system

$$\dot{x} = x(y^2 - y),$$
$$\dot{y} = x - y.$$

   (i). Sketch the nullclines and find the equilibria.
   (ii). Perform a linear stability analysis of each equilibrium.
   (iii). Carefully sketch the phase plane.

6. Consider the nonlinear system

$$\dot{x} = y,$$
$$\dot{y} = -y - \cos x,$$

   on the region $\{(x, y) \in \mathbb{R}^2 : -2\pi \leq x \leq 2\pi, \text{ and } -\pi \leq y \leq \pi\}$.

   (i). Sketch the nullclines and find the equilibria.
   (ii). Perform a linear stability analysis of each equilibrium.
   (iii). Carefully sketch the phase plane.

Problems 7 and 8 feature non-hyperbolic equilibria.

7. Consider the nonlinear system

$$\dot{x} = y,$$
$$\dot{y} = -x - y^3.$$

   (i). Sketch the nullclines and find the equilibria.
   (ii). Perform a linear stability analysis of each equilibrium.
   (iii). Will the linearization at $(0, 0)$ give a good approximation of the nonlinear flow? Explain.
   (iv). **Optional:** Prove that the equilibrium at the origin is asymptotically stable.
   Hint: switch to polar coordinates,

$$(x, y) = (r\cos\theta, r\sin\theta), \qquad r \in [0, \infty),\ \theta \in \mathbb{R},$$

   and show that $r \to 0$ as $t \to \infty$. Explain the geometric meaning of $\dot{\theta} \to -1$ as $t \to \infty$.
   (v). Carefully sketch the phase plane.
   (vi). Comment on the difference between the linearized flow at the origin and the actual nonlinear flow around the origin.

8. Consider the nonlinear system

$$\dot{x} = -x + xy,$$
$$\dot{y} = x^2 - y^2.$$

   (i). Sketch the nullclines and find the equilibria.
   (ii). For each equilibrium, write down the linearization and classify the equilibrium.
   (iii). Will the linearization at $(0, 0)$ give a good approximation of the nonlinear flow? Explain.
   (iv). Carefully sketch the phase plane.
   (v). Comment on the difference between the linearized flow at the origin and the actual nonlinear flow around the origin.
   (vi). **Optional:** show that the trajectories have the following asymptotic behaviour

$$y \to \pm\frac{x}{\sqrt{2}} \quad \text{as} \quad x \to \pm\infty.$$

   Hint: use the approximation $y \approx y - 1$ for $|y| \gg 1$.