BOSTON UNIVERSITY
MA 226 Section A

Spring Semester | Differential Equations | 2017

Additional Homework

1. Einstein's Theory of Relativity predicts the existence of black holes: regions in space from which nothing can escape, due to strong gravitational forces. The theory predicts that black holes will be formed when large stars collapse. However, Einstein's theory did not take into account quantum mechanical effects. In 1975, Stephen Hawking used quantum theory to show that a black hole should glow slightly; that is, it should radiate energy and particles in the same way that a heated object does. Assuming that nothing else falls into the black hole, this causes its mass M to decrease at the rate governed by the differential equation,

$$\frac{dM}{dt} = -\frac{\alpha}{M^2},$$

where t denotes time and α is a constant whose value is not yet known precisely.

   (i) Find the general solution $M(t)$ of this differential equation.
   (ii) Find the particular solution which satisfies the condition that the mass is M_0 when $t = 0$.
   (iii) How long does it take for a black hole which initially has mass M_0 to lose half of its mass? How long does it take for it to evaporate completely?

2. Consider the following ODE for v:

$$2xv\frac{dv}{dx} = 3v^2 - 4x^2, \tag{1}$$

which is neither separable nor linear.

   (i) Let $v(x) = xw(x)$. What is the ODE for $w(x)$?
   (ii) Let $u(x) = (w(x))^2$. Find the general solution $u(x)$.
   (iii) Hence, or otherwise, find the general solution $v(x)$ of (1).

3. **Challenging:** Solve the following (well-known) nonlinear ODEs by the methods indicated.

   (i) **Bernoulli Equation**

$$3x\frac{dy}{dx} + y + x^2y^4 = 0.$$

   Hint: let $w = 1/y^3$.

   (ii) **Riccati Equation**

$$\frac{dy}{dx} + xy^2 + \frac{3}{4x^3} = 0.$$

   Hint: let $y = \frac{1}{2x^2} + \frac{1}{w}$.

   (iii) **Clairaut Equation**

$$x\frac{dy}{dx} - y = \frac{1}{4}\left(\frac{dy}{dx}\right)^4.$$

   Hint: differentiate both sides.