BOSTON UNIVERSITY
MA 226 Section A

Spring Semester	Differential Equations	2017

Assignment

DUE DATE: April 10 before 5pm.

SUBMIT TO: my mailbox (*not my office*).

General guidelines.

- *Attach the Assignment Cover Sheet and this question sheet to the front of your assignment.*
- Only properly written assignments will be considered, i.e., what you submit is a document (like an essay) and not a draft written on scrap sheets. Poorly written (or unreadable) work will not be marked.
- Do not submit multiple versions of an answer. This will be interpreted as a draft and will not be considered for marking.
- The **bonus** questions are optional, and are not essential to completing the assignment.
- All diagrams must be *at least* $\frac{1}{3}$ of a page in size.

Consider the following model for the interactions between the populations N and M,

$$\begin{aligned}\frac{dN}{dt} &= rN\left(1 - \frac{N}{K}\right) - \alpha NM,\\ \frac{dM}{dt} &= sM\left(1 - \frac{M}{L}\right) - \beta NM,\end{aligned} \tag{1}$$

where r, s, K, L, α, and β are all positive parameters.

(i) Is the interaction predator-prey, competitive, or mutualistic? Explain your reasoning. **[2 MARKS]**

(ii) Briefly explain what each parameter represents. **[3 MARKS]**

(iii) Non-dimensionalize system (1) so that the dimensionless dynamics are given by

$$\begin{aligned}\frac{dx}{d\tau} &= x(1-x) - \mu_1 xy,\\ \frac{dy}{d\tau} &= \sigma y(1-y) - \mu_2 xy.\end{aligned}$$

What are the expressions for μ_1, μ_2, and σ in terms of the original parameters? **[2 MARKS]**

(iv) Determine the coordinates of all four equilibria. **[3 MARKS]**

Let (x_1, y_1), (x_2, y_2), and (x_3, y_3) denote the equilibria in which at least one of the coordinates is zero and let (x_4, y_4) be the equilibrium in which $x_4 \neq 0$ and $y_4 \neq 0$. This assignment is primarily concerned with how the model behaves under changes in the interaction rates, i.e., changes in μ_1 and μ_2.

(v) State the conditions on μ_1 and μ_2 such that (x_4, y_4) is in the biologically relevant domain. **[2 MARKS]**

(vi) Use linear stability analysis to classify (x_1, y_1), (x_2, y_2), and (x_3, y_3). **[5 MARKS]**

For parts (vii) – (ix) below, suppose that $\mu_1 < 1$ and $\mu_2 < \sigma$.

(vii) Prove that (x_4, y_4) is a stable node. **[3 MARKS]**

(viii) Determine the vector field along the x- and y-nullclines. **[2 MARKS]**

(ix) Carefully sketch the phase plane in the first quadrant $\{x \geq 0, y \geq 0\}$. **[7 MARKS]**

For part (x) below, suppose that $\mu_1 > 1$ and $\mu_2 < \sigma$.

(x) Carefully sketch the phase plane in the first quadrant $\{x \geq 0, y \geq 0\}$. **[3 MARKS]**

For part (xi) below, suppose that $\mu_1 < 1$ and $\mu_2 > \sigma$.

(xi) Carefully sketch the phase plane in the first quadrant $\{x \geq 0, y \geq 0\}$. **[3 MARKS]**

For parts (xii) – (xiii) below, suppose that $\mu_1 > 1$ and $\mu_2 > \sigma$.

(xii) Prove that (x_4, y_4) is a saddle. **[2 MARKS]**

(xiii) Carefully sketch the phase plane in the first quadrant $\{x \geq 0, y \geq 0\}$. **[3 MARKS]**

Finally, it is important to be able to explain what you have calculated. The explanation is as important as the calculation itself, since it indicates your understanding of your analysis.

(xiii) Interpret the phase planes from parts (ix) – (xiii) with respect to the original variables and parameters. At minimum, you should address the following points:

- Under what conditions do N and M stably coexist?
- When N and M do not stably coexist, what is the long-term behaviour of the system?
- Explain, in terms of the original parameters, why these different behaviours are observed.

[5 MARKS]

The remainder of the assignment is **bonus** material.

(xiv) Fix $\mu_2 < \sigma$. Explain what happens as μ_1 increases through 1. **[1 MARK]**

(xv) Fix $\mu_1 < 1$. Explain what happens as μ_2 increases through σ. **[1 MARK]**

(xvi) Sketch the *two-parameter plane* (i.e., plot μ_1 against μ_2) and show all bifurcation boundaries in the (μ_1, μ_2) plane. Give explicit formulae for the bifurcation boundaries. For this part, you may assume that $\sigma > 1$. You are not required to identify the types of bifurcations. **[3 MARKS]**