Boston University
MA 226 Section A

Spring Semester | Differential Equations | 2017

Midterm 1

NAME: Solutions BU ID:

Discussion Section (Day & Time):

General guidelines for the exam:

- Answer questions in the spaces provided.
- Diagrams should take up all of the given space.
- Bonus questions are *optional*. They are not essential to completing the exam.
- **Show all work, as partial credit will be given.**
- The time allowed is 45 minutes. You may not leave the room before the test is finished.
- The use of electronic devices (calculators, phones, laptops, etc) is strictly prohibited.
- Do not submit multiple versions of an answer. These will be not considered for marking.
- Cheating will not be tolerated.

Question	Value	Score
1	7	
2	7	
3	6	
4	4	
5	10	
6	10	
Total	44	
Bonuses	4	

Signature:

Question 1

Classify the following ODEs as separable or linear, and find the solution, x, as a function of t:

(i) $\dfrac{dx}{dt} = t + \dfrac{x}{t}$. LINEAR **[3 marks]**

$$\frac{dx}{dt} - \frac{x}{t} = t.$$

Integrating Factor, $I(t) = e^{\int \left(-\frac{1}{t}\right) dt} = \frac{1}{t}$.

Multiplying the ODE by $I(t)$:

$$\frac{1}{t}\frac{dx}{dt} - \frac{1}{t^2}x = 1$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{t}x\right) = 1$$

$$\Rightarrow \boxed{x = t^2 + Ct, \quad C \text{ arb. constant}}$$

(ii) $\dfrac{dx}{dt} = \dfrac{1}{tx + t + x + 1}$, $x(0) = 0$. SEPARABLE. **[4 marks]**

$$\frac{dx}{dt} = \frac{1}{(t+1)(x+1)}$$

$$\Rightarrow \int (x+1)\,dx = \int \frac{dt}{t+1}$$

$$\Rightarrow \frac{1}{2}(x+1)^2 = \ln(t+1) + C.$$

Initial Condition, $x(0) = 0$:

$$\frac{1}{2} = \ln 1 + C \Rightarrow C = \frac{1}{2}$$

Substituting into the solution:

$$\frac{1}{2}(x+1)^2 = \ln(t+1) + \frac{1}{2}$$

$$\Rightarrow x = -1 \pm \sqrt{\ln(t+1)^2 + 1}$$

Since the solution of this IVP is unique, the correct solution is

$$\boxed{x = -1 + \sqrt{\ln(t+1)^2 + 1}}$$

Question 2

Consider the ODE

$$t\frac{dx}{dt} - x - t^2 \ln t = 0. \quad (1)$$

(i). Show that $x = t^2 \ln t - t^2$ is a solution of (1). **[1 mark]**

$t\frac{dx}{dt} - x - t^2 \ln t = t\left(t + 2t\ln t - 2t\right) - \left(t^2 \ln t - t^2\right) - t^2 \ln t$

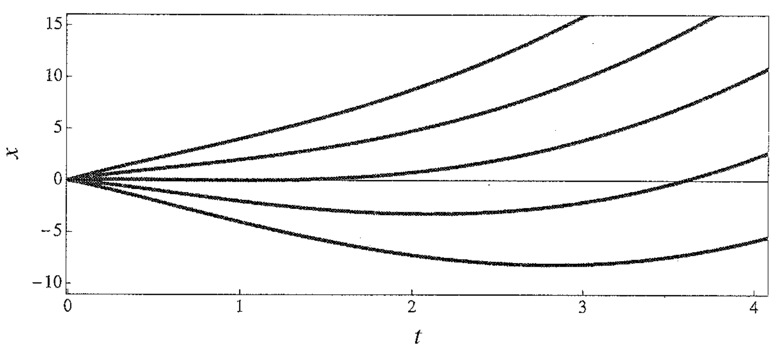
$= 2t^2 \ln t - t^2 - t^2 \ln t + t^2 - t^2 \ln t = 0.$

(ii). Consider (1) with the initial condition $x(1) = 0$. Does a unique solution of this initial value problem exist? Explain why or why not. **[2 marks]**

The ODE is $\frac{dx}{dt} = \frac{x}{t} + t \ln t =: f(x,t)$.

Note that $f(x,t) = \frac{x}{t} + t\ln t$ and $\frac{\partial f}{\partial x} = \frac{1}{t}$ are both continuous in a neighbourhood of the initial condition at $(t,x) = (1,0)$. Thus, by the existence and uniqueness theorems, this initial value problem has a unique solution.

(iii). The diagram below shows solution curves of (1) corresponding to different initial conditions.



Is there any issue at the origin? Explain why or why not. **[2 marks]**

Both f and $\frac{\partial f}{\partial x}$ are not continuous at $t=0$.

Thus, the existence and uniqueness theorems provide no information about solutions near the origin.

Hence, the non-uniqueness of solutions at the origin is a non-issue.

(iv). Explain why the diagram and your answer in (ii) do not contradict each other. **[2 marks]**

The uniqueness of the solution in (ii) is only guaranteed on some interval around the initial condition $x(1) = 0$.

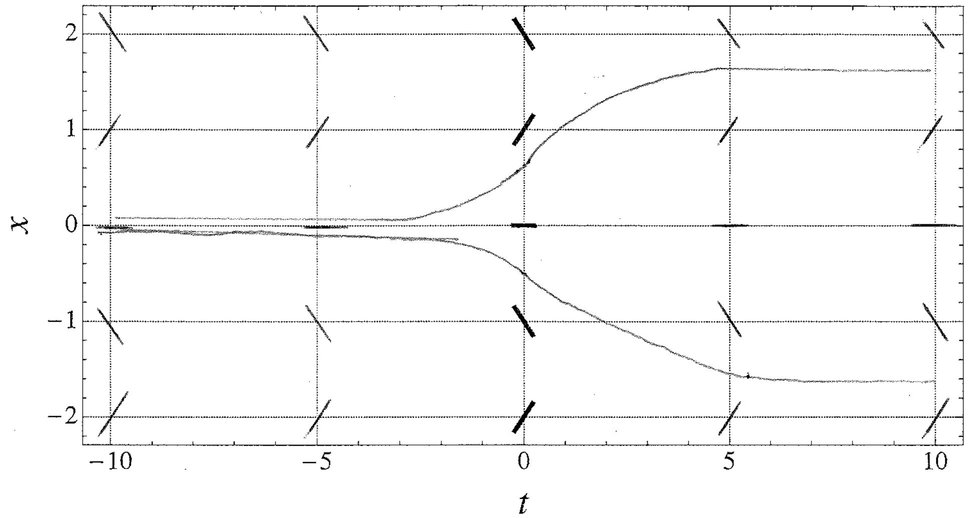
Thus, we cannot expect uniqueness to hold far away from the initial condition, ie, at the origin.

Question 3

The diagram below shows the slope field of the ODE

$$\frac{dx}{dt} = f(x),$$

for a specific continuous function f, along the grid-line $\{t = 0\}$.



(i) On the diagram above, sketch the slope field at the given grid points. **[2 marks]**

Equation is autonomous $\Rightarrow$ slope field constant along horizontal lines

(ii) In the region of the (x, t) plane shown above, how many equilibria does this equation have? Explain your reasoning. **[2 marks]**

Slope is horizontal along $x = 0$, i.e. $\frac{dx}{dt} = 0$ along $x = 0 \Rightarrow x = 0$ is an eqm.

Observe: $\frac{dx}{dt} > 0$ along $x = 1$ and $\frac{dx}{dt} < 0$ along $x = 2$.

$\Rightarrow$ By continuity, there exists an eqm between $x = 1$ and $x = 2$.

Similarly, there exists an eqm between $x = -2$ and $x = -1$.

(iii) Sketch the solution with initial condition $x(0) = x_0 \in (0, 1)$, for $-10 \le t \le 10$. **[1 mark]**

(iv) Sketch the solution with initial condition $x(0) = x_0 \in (-1, 0)$, for $-10 \le t \le 10$. **[1 mark]**

Question 4

Consider the general linear ODE
$$\frac{dx}{dt} + p(t)\,x = q(t), \tag{2}$$

where $p(t)$ and $q(t)$ are continuous functions. Suppose $x_H(t)$ is a solution of the homogeneous problem and $x_p(t)$ is a solution of the inhomogeneous problem.

(i) Carefully prove the linearity principle for (2). That is, show that $x_p(t) + Cx_H(t)$ is a solution of (2) for any constant C. **[3 marks]**

$$\frac{d}{dt}(x_p + Cx_H) + p(t)(x_p + Cx_H) = \left(\frac{dx_p}{dt} + px_p\right) + C\left(\frac{dx_H}{dt} + px_H\right)$$
$$= q + C(0),$$
$$= q,$$

since x_p is a sol^{n} of the inhomogeneous equation, and x_H is a sol^{n} of the homogeneous equation.

Thus, $x_p + Cx_H$ is a sol^{n} of (2).

(ii) Explain why $x_p(t) + Cx_H(t)$ is the general solution of (2). **[1 mark]**

(2) is first-order and $x_p + Cx_H$ has one arbitrary constant.

(iii) **Bonus:** Derive formulas for $x_H(t)$ and $x_P(t)$. Show all work. **[2 marks]**

Integrating Factor, $I(t) = \exp\left(\int p(t)\,dt\right)$

Note: $\frac{dI}{dt} = p(t)\,I(t)$ by F.T.C.

Multiplying (2) by $I(t)$:

$$I\frac{dx}{dt} + Ipx = Iq.$$
$$\Rightarrow I\frac{dx}{dt} + \frac{dI}{dt}x = Iq,$$
$$\Rightarrow \frac{d}{dt}(Ix) = Iq.$$
$$\Rightarrow x(t) = \frac{1}{I}\int Iq\,dt + \frac{C}{I}$$

Thus, $x_H(t) = \frac{1}{I} = \exp\left(-\int p(t)\,dt\right)$ and $x_p = \frac{1}{I}\int Iq\,dt = e^{-\int p\,dt}\int q e^{\int p\,dt}\,dt.$

Question 5

Consider the harvesting model

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - EP, \tag{☼}$$

where the population, P, is harvested at constant effort E at a rate EP. The parameters k, N and E are non-negative. The population is measured in kilograms and time is measured in seconds. The rescaling, $P = Nx$ and $t = \frac{1}{k}\tau$, yields the dimensionless harvesting model

$$\frac{dx}{d\tau} = x(1-x) - \mu x, \tag{☺}$$

where x and τ are the dimensionless population and time, respectively.

(i). What is μ in terms of k, N and E? Explain why μ is dimensionless. **[2 marks]**

$$\mu = \frac{E}{k}. \Rightarrow [\mu] = \frac{[E]}{[k]} = \frac{\frac{1}{\text{seconds}}}{\frac{1}{\text{seconds}}} = 1.$$

i.e., μ is dimensionless.

(ii). Find the equilibria of (☺) and determine their stability. **[3 marks]**

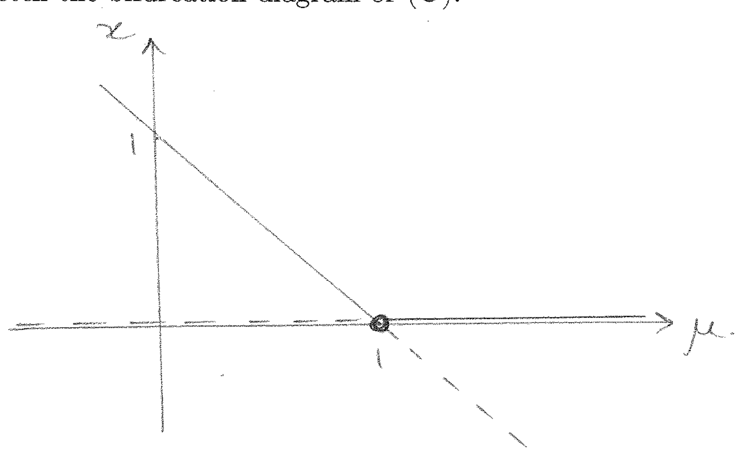
Let $f(x,\mu) = x(1-x-\mu)$. Then $\frac{\partial f}{\partial x} = 1-\mu-2x$

Equilibria: $\dot{x} = 0 \Rightarrow x = 0$ OR $x = 1-\mu$.

Stability: Along $x=0$, $\frac{\partial f}{\partial x} = 1-\mu \begin{cases} >0 \text{ for } \mu<1 \Rightarrow \text{UNSTABLE} \\ <0 \text{ for } \mu>1. \Rightarrow \text{STABLE} \end{cases}$

Along $x = 1-\mu$, $\frac{\partial f}{\partial x} = \mu - 1 \begin{cases} <0 \text{ for } \mu<1 \Rightarrow \text{STABLE} \\ >0 \text{ for } \mu>1. \Rightarrow \text{UNSTABLE} \end{cases}$

(iii). Sketch the bifurcation diagram of (☺). **[3 marks]**



(iv). Give a biological interpretation of your bifurcation diagram in (iii) with respect to the original variables and parameters in (☼). **[2 marks]**

For $\mu \geq 1$, the population always becomes extinct.

That is, for $\frac{E}{k} \geq 1$, the harvesting destroys the population.

OR for $E \geq k$, the harvesting rate overwhelms the population growth rate and nothing survives.

Question 6

Consider the autonomous ODE

$$\frac{dx}{dt} = x^2 - \mu x + 1,$$

where $\mu \in \mathbb{R}$ is a parameter.

(i) Find all bifurcation points. **[2 marks]**

Let $f(x, \mu) = x^2 - \mu x + 1$. Then $\frac{\partial f}{\partial x} = 2x - \mu$.

Bifurcation points occur when $f = 0$ and $\frac{\partial f}{\partial x} = 0$:

$$\left.\begin{array}{l} x^2 - \mu x + 1 = 0 \\ 2x - \mu = 0 \end{array}\right\} \Rightarrow \boxed{(x_1, \mu_1) = (-1, -2) \text{ and } (x_2, \mu_2) = (1, 2)}.$$

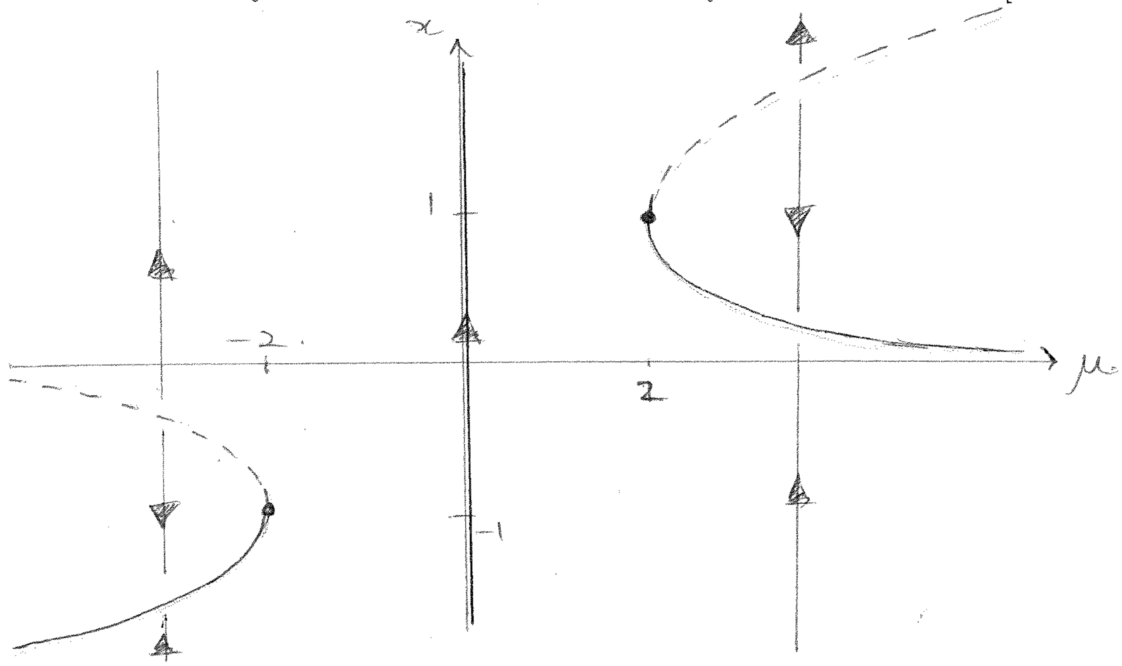
(ii) By computing higher order derivatives, classify the bifurcation points in (i). **[2 marks]**

$\frac{\partial f}{\partial \mu} = -x \neq 0 \quad \forall\, x \neq 0$ and $\mu \in \mathbb{R}$.

$\frac{\partial^2 f}{\partial x^2} = 2 \neq 0 \quad \forall\, x$ and μ.

$\Rightarrow (x_1, \mu_1)$ and (x_2, μ_2) are saddle-node bifurcations.

(iii) Hence, or otherwise, *carefully* sketch the bifurcation diagram. The reverse side of this page has been left blank for you to do additional calculations if you need. **[4 marks]**



(iv) On your diagram above, sketch the phase lines for $\mu = -4$, $\mu = 0$, and $\mu = 4$. **[2 marks]**