MA226 HW WEEK 03

HANDOUT 30/01/17, QUESTION 3(iii)

Clairaut Equation: $x\frac{dy}{dx} - y = \frac{1}{4}\left(\frac{dy}{dx}\right)^4$.

Differentiating both sides with respect to x:

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{dy}{dx} = \left(\frac{dy}{dx}\right)^3 \frac{d^2y}{dx^2}.$$

$$\Rightarrow \frac{d^2y}{dx^2}\left(\left(\frac{dy}{dx}\right)^3 - x\right) = 0.$$

There are 2 possibilities: either $\frac{d^2y}{dx^2} = 0$ OR $\left(\frac{dy}{dx}\right)^3 - x = 0$.

CASE 1: $\frac{d^2y}{dx^2} = 0 \Rightarrow y = Ax + B$, A, B arb. constants.

Note that the Clairaut equation is first-order. Thus, its general solution can only have one arbitrary constant. This implies that there must be a relationship between A and B. To determine that relationship, substitute $y = Ax + B$ into the Clairaut equation:

$$x \cdot A - (Ax + B) = \frac{1}{4}(A)^4$$

$$\Rightarrow B = -\frac{1}{4}A^4.$$

Thus, $y = Ax - \frac{1}{4}A^4$ is the general solution.

CASE 2: $\left(\frac{dy}{dx}\right)^3 = x \Rightarrow \frac{dy}{dx} = x^{1/3}$.

$$\Rightarrow y = \frac{3}{4}x^{4/3} + C, \quad C \text{ arb. constant}.$$

Thus, the Clairaut equation has 2 types of solutions: the straight lines $y = Ax - \frac{1}{4}A^4$ and the power laws $y = \frac{3}{4}x^{4/3} + C$.

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SECTION 1.5, QUESTION 9

$$\frac{dy}{dt} = -y^2 + y + 2yt^2 + 2t - t^2 - t^4 =: f(y,t).$$

a). Let $y_1 = t^2$. Then $\frac{dy_1}{dt} = 2t$.

And $f(y_1, t) = -t^4 + t^2 + 2t^4 + 2t - t^2 - t^4 = 2t$.

Thus, $\frac{dy_1}{dt} = f(y_1, t)$.

$\Rightarrow y_1 = t^2$ is a solution of the ODE.

Let $y_2 = t^2+1$. Then $\frac{dy_2}{dt} = 2t$.

And
$$\begin{aligned} f(y_2,t) &= -(t^2+1)^2 + (t^2+1) + 2(t^2+1)t^2 + 2t - t^2 - t^4 \\ &= -t^4 - 2t^2 - 1 + t^2 + 1 + 2t^4 + 2t^2 + 2t - t^2 - t^4 \\ &= 2t \end{aligned}$$

Thus, $\frac{dy_2}{dt} = f(y_2, t)$

$\Rightarrow$ $y_2 = t^2+1$ is a solution of the ODE.

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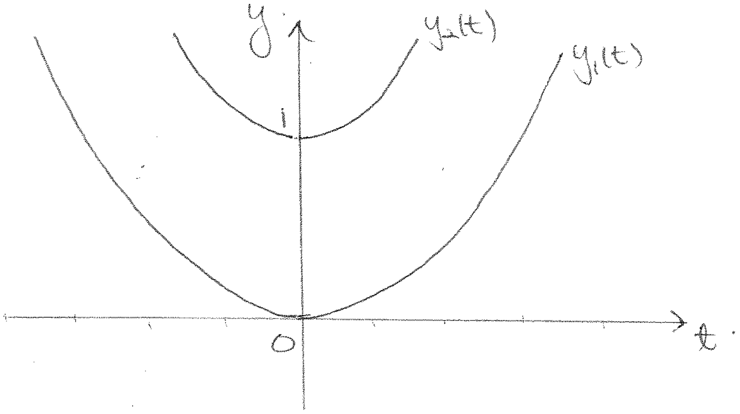
b). NOTE: $f(y,t) = -y^2 + y + 2yt^2 + 2t - t^2 - t^4$ is continuous for all t and y.

And $\frac{\partial f}{\partial y} = -2y + 1 + 2t^2$ is continuous for all t and y.

Thus, the solutions of $\frac{dy}{dt} = f(y,t)$ obey the uniqueness theorem.



Also note that $y_1(0) = 0$ and $y_2(0) = 1$.

Thus, by the uniqueness theorem, all solutions, y, with $y(0) \in (0,1)$ are bounded by $y_1(t)$ and $y_2(t)$ for all t.

HANDOUT 03/02/17, QUESTION 1

$$\varphi_1(t) \leq c(t) + \int_0^t \varphi_1(s)\,\varphi_2(s)\,ds \quad \forall\, t \in [0,T].$$

(i). $G(t) := c(t) + \int_0^t \varphi_1(s)\,\varphi_2(s)\,ds.$

$$\Rightarrow \varphi_1(t) \leq G(t).$$

(ii). Differentiating G with respect to t:

$$\frac{dG}{dt} = \frac{dc}{dt} + \varphi_1(t)\,\varphi_2(t), \text{ by F.T.C.}$$

$$\Rightarrow \frac{dG}{dt} \leq \frac{dc}{dt} + G(t)\,\varphi_2(t) \text{ by (i) above.}$$

$$\Rightarrow \frac{dG}{dt} - \varphi_2(t)\,G \leq \frac{dc}{dt}$$

Integrating Factor, $I(t) = \exp\left(-\int_0^t \varphi_2(s)\,ds\right)$.

Multiplying the inequality by $I(t)$ gives

$$e^{-\int_0^t \varphi_2 ds}\frac{dG}{dt} - \varphi_2 e^{-\int_0^t \varphi_2 ds} G \leq \frac{dc}{dt} e^{-\int_0^t \varphi_2 ds}.$$

$$\Rightarrow \frac{d}{dt}\left(e^{-\int_0^t \varphi_2(s)ds} G\right) \leq \frac{dc}{dt} e^{-\int_0^t \varphi_2(s)\,ds} \qquad \forall\ t \in [0,T].$$

(iii). Since $\varphi_2 : [0,T] \to [0,\infty)$, φ_2 is always non-negative.

Thus, $-\int_0^t \varphi_2(s)\,ds$ is always non-positive. That is,

$$-\infty < -\int_0^t \varphi_2(s)\,ds \leq 0.$$

$$\Rightarrow 0 < e^{-\int_0^t \varphi_2(s)\,ds} \leq 1.$$

Applying this inequality to the result in (ii), we have

$$\frac{d}{dt}\left(G \cdot e^{-\int_0^t \varphi_2(s)\,ds}\right) \leq \frac{dc}{dt} \quad \forall\ t \in [0,T].$$

(iv). $$\frac{d}{dt}\left(G \cdot e^{-\int_0^t \varphi_2(s)\,ds}\right) \leq \frac{dc}{dt} \quad \forall\ t \in [0,T].$$

Integrating with respect to t from $t=0$ up to $t=t$:

$$\Rightarrow \left(G\, e^{-\int_0^t \varphi_2(s)\,ds}\right)\Big|_0^t \leq c\,\Big|_0^t \qquad \forall\ t \in [0,T].$$

$$\Rightarrow G(t)\, e^{-\int_0^t \varphi_2(s)ds} - G(0) \leq c(t) - c(0).$$

Note that $G(0) = c(0) + \int_0^0 \varphi_1(s)\varphi_2(s)\,ds = c(0)$.

$$\Rightarrow G(t)\, e^{-\int_0^t \varphi_2(s)\,ds} \leq c(t)$$

$$\Rightarrow G(t) \leq c(t) \exp\left\{\int_0^t \varphi_2(s)\,ds\right\}$$

Recall from (i) that $\varphi_1(t) \leq G(t)$. Thus,

$$\varphi_1(t) \leq c(t) \exp\left\{\int_0^t \varphi_2(s)\,ds\right\}.$$
