MA 226 HW WEEK 09

HANDOUT

(i). $\sin^2(A) = \frac{1}{-4}\left(\exp(iA)\exp(iA) - \exp(iA)\exp(-iA) - \exp(-iA)\exp(iA) + \exp(-iA)\exp(-iA)\right)$

$\Rightarrow \sin^2(A) = -\frac{1}{4}\left(\exp(iA)\exp(iA) + \exp(-iA)\exp(-iA) - 2I\right)$

$\cos^2(A) = \frac{1}{4}\left(\exp(iA)\exp(iA) + \exp(-iA)\exp(-iA) + 2I\right)$

$\Rightarrow \sin^2(A) + \cos^2(A) = \frac{1}{4}\left(2I + 2I\right) = I.$

(ii): In lectures, we showed that $\frac{d}{dt}\exp(At) = A\exp(At)$.

$$\frac{d}{dt}\left(\sin(At)\right) = \frac{1}{2i}\frac{d}{dt}\left(\exp(iAt) - \exp(-iAt)\right)$$

$$= \frac{1}{2i}\left(iA\exp(iAt) + iA\exp(-iAt)\right)$$

$$= \frac{1}{2}A\left(\exp(iAt) + \exp(-iAt)\right)$$

$$= A\cos(At).$$

Similarly, $\frac{d}{dt}\left(\cos(At)\right) = -A\sin(At)$.

(iii). $x = \frac{\sqrt{H}}{a}\sin(at+c) \Rightarrow \dot{x} = \sqrt{H}\cos(at+c)$

$$\Rightarrow \ddot{x} = -a\sqrt{H}\sin(at+c)$$

$$= -a^2\frac{\sqrt{H}}{a}\sin(at+c)$$

$$= -a^2x$$

$$\Rightarrow \ddot{x} + a^2x = 0$$

Thus, $x = \frac{\sqrt{H}}{a}\sin(at+c)$ is a solution of $\ddot{x} + a^2x = 0$.

Moreover, it is the general solution since there are 2 arbitrary constants.

(iv). $\underline{x} = \cos(At)\underline{x_0} + A^{-1}\sin(At)\underline{v_0}$

$\Rightarrow \dot{\underline{x}} = -A\sin(At)\underline{x_0} + \cos(At)\underline{v_0}$.

$\Rightarrow \ddot{\underline{x}} = -A^2\cos(At)\underline{x_0} - A\sin(At)\underline{v_0} = -A^2\left(\cos(At)\underline{x_0} + A^{-1}\sin(At)\underline{v_0}\right) = -A^2\underline{x}$.

$\Rightarrow \underline{x}$ is a solution of $\ddot{\underline{x}} + A^2\underline{x} = \underline{0}$.

It is the general solution since there are 4 arbitrary constants (2 from $\underline{x_0}$ and 2 from $\underline{v_0}$).

SECTION 3.3, QUESTION 27

$$\frac{dy}{dt} = \begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix} y$$

a). Equilibrium at $\underline{y} = \underline{0}$.

Stability: $D\underline{f}(\underline{y}) = \begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix} \Rightarrow D\underline{f}(\underline{0}) = \begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}$.

$\Rightarrow$ Eigenvalues: $\lambda_1 = -2$, $\lambda_2 = 2$ ... SADDLE.

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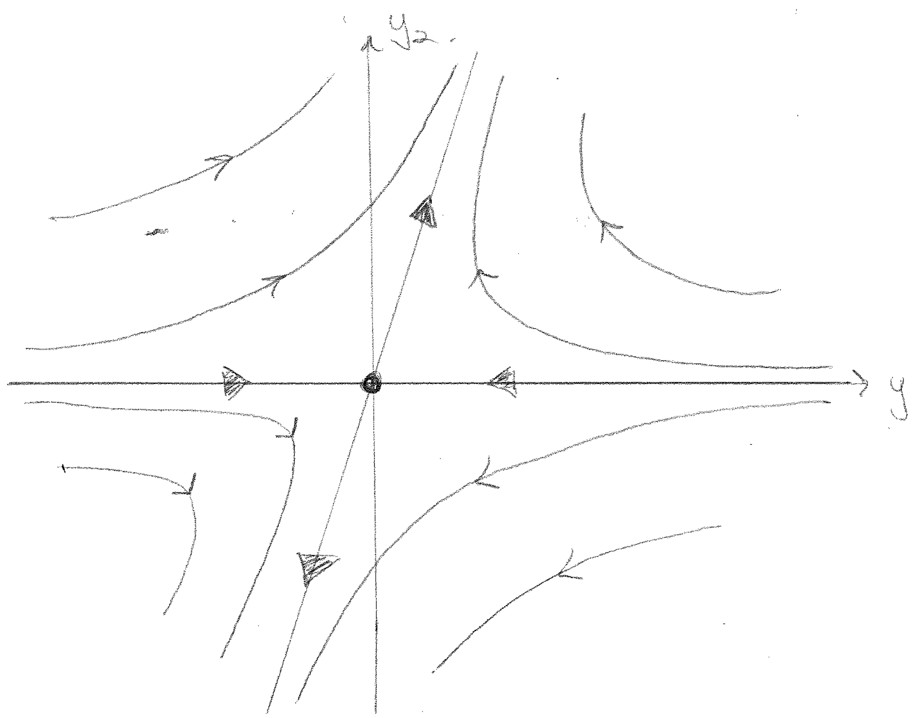
b). Eigenvector for $\lambda_1 = -2$

$D\underline{f}(\underline{0}) - \lambda_1 I = \begin{pmatrix} 0 & 1 \\ 0 & 4 \end{pmatrix} \Rightarrow v_2 = 0 \Rightarrow \underline{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Eigenvector for $\lambda_2 = 2$

$D\underline{f}(\underline{0}) - \lambda_2 I = \begin{pmatrix} -4 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow -4v_1 + v_2 = 0 \Rightarrow \underline{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$



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d). Let $y_1(t)$ be the solution with $y_1(0) = \begin{pmatrix}1\\ \frac{1}{100}\end{pmatrix}$, and let $y_2(t)$ be the solution with $y_2(0)=\begin{pmatrix}1\\ -\frac{1}{100}\end{pmatrix}$

The G.S. is $y(t) = C_1\begin{pmatrix}1\\0\end{pmatrix}e^{-2t} + C_2\begin{pmatrix}1\\4\end{pmatrix}e^{2t}$.

$\underline{y_1(0) = \begin{pmatrix}1\\ \frac{1}{100}\end{pmatrix}} \Rightarrow \begin{pmatrix}1\\ \frac{1}{100}\end{pmatrix} = C_1\begin{pmatrix}1\\0\end{pmatrix} + C_2\begin{pmatrix}1\\4\end{pmatrix} \Rightarrow \begin{pmatrix}C_1\\C_2\end{pmatrix} = \frac{1}{4}\begin{pmatrix}4 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ \frac{1}{100}\end{pmatrix} = \frac{1}{4}\begin{pmatrix}4-\frac{1}{100}\\ \frac{1}{100}\end{pmatrix}$

$$\Rightarrow y_1(t) = \left(1-\frac{1}{400}\right)\begin{pmatrix}1\\0\end{pmatrix}e^{-2t} + \frac{1}{400}\begin{pmatrix}1\\4\end{pmatrix}e^{2t}.$$

$\underline{y_2(0) = \begin{pmatrix}1\\ -\frac{1}{100}\end{pmatrix}} \Rightarrow \begin{pmatrix}1\\ -\frac{1}{100}\end{pmatrix} = C_1\begin{pmatrix}1\\0\end{pmatrix} + C_2\begin{pmatrix}1\\4\end{pmatrix} \Rightarrow \begin{pmatrix}C_1\\C_2\end{pmatrix} = \frac{1}{4}\begin{pmatrix}4 & -1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ -\frac{1}{100}\end{pmatrix} = \frac{1}{4}\begin{pmatrix}4+\frac{1}{100}\\ -\frac{1}{100}\end{pmatrix}$

$$\Rightarrow y_2(t) = \left(1+\frac{1}{400}\right)\begin{pmatrix}1\\0\end{pmatrix}e^{-2t} \cancel{+} -\frac{1}{400}\begin{pmatrix}1\\4\end{pmatrix}e^{2t}.$$

Solutions y_1 and y_2 will be 1 unit apart when

$$\|y_1 - y_2\| = 1.$$

$$\Rightarrow \left\| \frac{1}{200}\begin{pmatrix}1\\0\end{pmatrix}e^{-2t} - \frac{1}{200}\begin{pmatrix}1\\4\end{pmatrix}e^{2t} \right\| = 1.$$

$$\Rightarrow \left\| \begin{pmatrix}1\\0\end{pmatrix}e^{-2t} - \begin{pmatrix}1\\4\end{pmatrix}e^{2t} \right\| = 200.$$

$$\Rightarrow \left(e^{-2t} - e^{2t}\right)^2 + \left(-4e^{2t}\right)^2 = 200^2.$$

$$\Rightarrow e^{-4t} + e^{4t} - 2 + 16e^{4t} = 200^2.$$

$$\Rightarrow 17e^{4t} - (2+200^2) + e^{-4t} = 0.$$

$$\Rightarrow 17e^{8t} - (2+200^2)e^{4t} + 1 = 0.$$

$$\Rightarrow e^{4t} = \frac{1}{34}\left(2+200^2 \pm \sqrt{(2+200^2)^2 - 68}\right)$$

$$\Rightarrow t = \frac{1}{4}\log\left(\frac{2+200^2 \pm \sqrt{(2+200^2)^2-68}}{34}\right)$$

We can eliminate the negative root as a possibility since that would give $t < 0$. Thus,

$$t = \frac{1}{4}\log\left(\frac{2+200^2 + \sqrt{(2+200^2)^2-68}}{34}\right)$$