MA 226 HW WEEK 11

HANDOUT, QUESTION 8

$$\dot{x} = -x + xy$$
$$\dot{y} = x^2 - y^2.$$

(i). x-nullcline: $\dot{x} = 0 \Rightarrow x(y-1) = 0 \Rightarrow x = 0$ OR $y = 1$.

y-nullcline: $\dot{y} = 0 \Rightarrow x^2 - y^2 = 0 \Rightarrow y = \pm x$.

Equilibria: $(x_0, y_0) = (0,0)$, $(x_1, y_1) = (-1, 1)$, $(x_2, y_2) = (1,1)$.

(ii). Stability: $DF(x,y) = \begin{pmatrix} y-1 & x \\ 2x & -2y \end{pmatrix}$

At $(x_0, y_0) = (0,0)$, $\dot{\underline{u}} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \underline{u}$.

$(\lambda_1, \underline{v_1}) = \left(-1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$, $(\lambda_2, \underline{v_2}) = \left(0, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$ ... SADDLE-NODE.

At $(x_1, y_1) = (-1, 1)$; $\dot{\underline{u}} = \begin{pmatrix} 0 & -1 \\ -2 & -2 \end{pmatrix} \underline{u}$.

$(\lambda_1, \underline{v_1}) = \left(-1-\sqrt{3}, \begin{pmatrix} 1 \\ 1+\sqrt{3} \end{pmatrix}\right)$, $(\lambda_2, \underline{v_2}) = \left(-1+\sqrt{3}, \begin{pmatrix} 1 \\ 1-\sqrt{3} \end{pmatrix}\right)$ ... SADDLE.

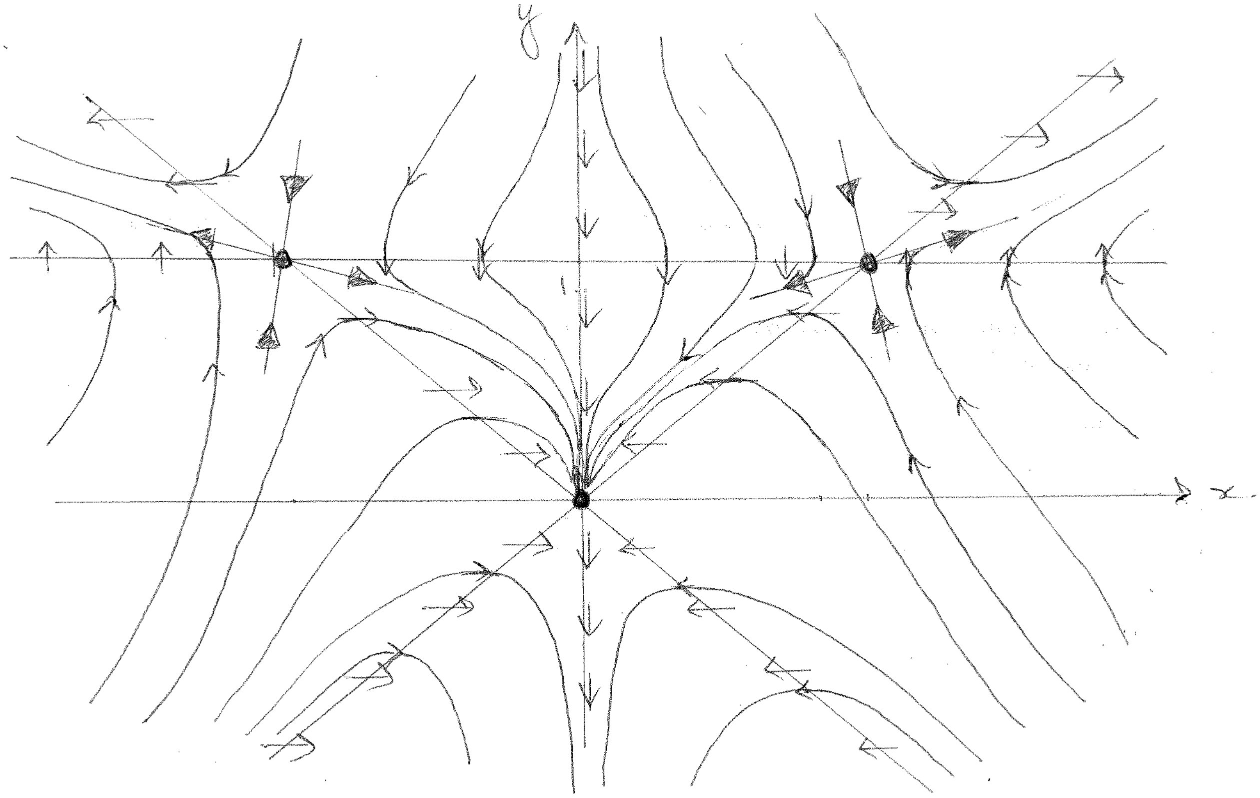
At $(x_2, y_2) = (1,1)$, $\dot{\underline{u}} = \begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix} \underline{u}$

$(\lambda_1, \underline{v_1}) = \left(-1-\sqrt{3}, \begin{pmatrix} 1 \\ -(1+\sqrt{3}) \end{pmatrix}\right)$, $(\lambda_2, \underline{v_2}) = \left(-1+\sqrt{3}, \begin{pmatrix} 1 \\ \sqrt{3}-1 \end{pmatrix}\right)$ ... SADDLE

(iii). $(x_0, y_0) = (0,0)$ is a SADDLE-NODE, i.e., (x_0, y_0) is non-hyperbolic.

Thus, the Hartman-Grobman Theorem does not apply and the linearization is not likely to give a good approximation of the nonlinear flow.

(iv).



NOTE: This system is symmetric about the y-axis.

(v). In a neighbourhood of (x_0, y_0):

- Nonlinear flow has one eqm, which is stable for $y > 0$ and unstable for $y < 0$.
- Linearization predicts line of equilibria, passing through $(0,0)$.

(vi). Trajectories of $\begin{cases} \dot{x} = -x + xy \\ \dot{y} = x^2 - y^2 \end{cases}$ are also solutions of $\frac{dy}{dx} = \frac{x^2 - y^2}{x(y-1)}$.

For $|y| >> 1$, $\frac{dy}{dx} \approx \frac{x^2 - y^2}{xy}$

$$\Rightarrow y \frac{dy}{dx} \approx x - \frac{1}{x} y^2$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{2}y^2\right) \approx x - \frac{2}{x} \cdot \left(\frac{1}{2}y^2\right)$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{2}y^2\right) + \frac{2}{x} \cdot \left(\frac{1}{2}y^2\right) \approx x$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{2}y^2 \cdot x^2\right) \approx x^3$$

$$\Rightarrow \frac{1}{2}x^2 y^2 \approx \frac{1}{4}x^4 + C$$

$$\Rightarrow y^2 \approx \frac{1}{2}x^2 + \frac{C}{x^2}$$

$$\lim_{x \to \pm\infty} y^2 \approx \frac{1}{2}x^2 \Rightarrow \lim_{x \to \pm\infty} y = \pm \frac{x}{\sqrt{2}}$$