SECTION 6.1, QUESTION 15

$$\frac{dy}{dt} = -y + e^{-2t}, \quad y(0) = 2.$$

Let $F(s) = \mathcal{L}\{y\}$. Transforming the ODE:

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = -\mathcal{L}\{y\} + \mathcal{L}\{e^{-2t}\}$$

$$\Rightarrow sF(s) - 2 = -F(s) + \frac{1}{s+2}$$

$$\Rightarrow (s+1)F(s) = \frac{1}{s+2} + 2.$$

$$\Rightarrow F(s) = \frac{1}{(s+1)(s+2)} + \frac{2}{s+2}$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}.$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}.$$

$$= e^{-t} - e^{-2t} + 2e^{-2t}.$$

$$\Rightarrow x(t) = e^{-t} + e^{-2t}.$$

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SECTION 6.2, QUESTION 16

$$\mathcal{L}\{f\} = \int_0^\infty f(t)e^{-st}\,dt$$

$$= \int_0^T f(t)e^{-st}\,dt + \int_T^{2T} f(t)e^{-st}\,dt + \dots + \int_{nT}^{(n+1)T} f(t)e^{-st}\,dt + \dots$$

$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} f(t)e^{-st}\,dt.$$

Let $t = u + nT$. Then $dt = du$.

$t = nT \Rightarrow u = 0$

$t = (n+1)T \Rightarrow u = T.$

$$\Rightarrow \mathcal{L}\{f\} = \sum_{n=0}^{\infty} \int_0^T f(u+nT)e^{-s(u+nT)}\,du.$$

$$= \sum_{n=0}^{\infty} e^{-Tns} \int_0^T f(u)e^{-su}\,du, \quad \text{since } f(u+T) = f(u)$$

Note that the integral is independent of the summation index n.

Thus, $\mathcal{L}\{f\} = \left(\sum_{n=0}^{\infty} e^{-Tsn} \right) \int_0^T f(t) e^{-st} \, dt$ (replacing u by t).

Now, $\sum_{n=0}^{\infty} e^{-Tsn} = 1 + e^{-Ts} + e^{-2Ts} + \dots$

$$\Rightarrow \sum_{n=0}^{\infty} e^{-Tsn} - 1 = e^{-Ts} + e^{-2Ts} + \dots = e^{-Ts}\left(1 + e^{-Ts} + \dots\right) = e^{-Ts} \sum_{n=0}^{\infty} e^{-Tsn}.$$

$$\Rightarrow \sum_{n=0}^{\infty} e^{-Tsn} = \frac{1}{1-e^{-Ts}}$$ (only works if $|e^{-Ts}| < 1$).

$$\Rightarrow \mathcal{L}\{f\} = \frac{1}{1-e^{-Ts}} \int_0^T f(t) e^{-st} \, dt.$$

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SECTION 6.2, QUESTION 14

$$\frac{dy}{dt} = -2y + H(t-2)e^{-t}, \quad y(0) = 3.$$

Let $F(s) = \mathcal{L}\{y\}$. Transforming the ODE:

$$s \cdot F - 3 = -2F + \mathcal{L}\{H(t-2)e^{-t}\}.$$

$$\Rightarrow (s+2)F = 3 + \mathcal{L}\{H(t-2)e^{-(t-2)-2}\}.$$

$$= 3 + e^{-2}\mathcal{L}\{H(t-2)e^{-(t-2)}\}.$$

$$= 3 + e^{-2} \cdot e^{-2s} \mathcal{L}\{e^{-t}\}$$ by the Shift Theorem.

$$= 3 + \frac{1}{e^2} \cdot \frac{e^{-2s}}{s+1}$$

$$\Rightarrow F = \frac{3}{s+2} + \frac{e^{-2s}}{e^2} \frac{1}{(s+1)(s+2)}$$

$$\Rightarrow y(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{1}{e^2}\mathcal{L}^{-1}\left\{e^{-2s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right)\right\}.$$

$$= 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{1}{e^2}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+1}\right\} - \frac{1}{e^2}\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s+2}\right\}.$$

$$\Rightarrow y(t) = 3e^{-2t} + \frac{1}{e^2}H(t-2)e^{-(t-2)} - \frac{1}{e^2}H(t-2)e^{-2(t-2)}$$ by the Shift Thm.

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