MA 123
Calculus I
Practice Final Exam
Answer Key

Note: Be aware that there may be *more* than one method to solving any one question. Keep in mind that the beauty in math is that you can often obtain the same answer from more than one approach. For this reason, the answers provided in this key are left in their most simplified form. Unless otherwise noted, the answers do not *need* to be simplified, yet simplification may be the key to dissecting a seemingly unworkable function.

1.) Evaluate the following indefinite integrals.

a.) $\int \frac{\sin\vartheta + \sin\vartheta \tan^2\vartheta}{\sec^2\vartheta}\, d\vartheta$

→ $\int \frac{\sin\vartheta(1+\tan^2\vartheta)}{\sec^2\vartheta}\, d\vartheta$; Factor out $\sin\theta$

→ $\int \frac{\sin\vartheta \sec^2\vartheta}{\sec^2\vartheta}\, d\vartheta$; $1+\tan^2\theta = \sec^2\theta$

→ $\int \sin\vartheta \; d\vartheta$; Cancel out $\sec^2\theta$

→ $-\cos\vartheta \; + C$ //

b.) $\int \frac{1}{(5t+4)^{2.7}}\, dt$

Let $u = 5t+4$. Then $du = 5dt$ and $dt = \frac{1}{5}du$, so

→ $\int \frac{1}{(5t+4)^{2.7}}\, dt = \int \frac{1}{(u)^{2.7}}\left(\frac{1}{5}du\right)$; Substitute in *u* and *du*

$= \frac{1}{5}\cdot\frac{1}{-1.7}u^{-1.7} + C$; Evaluate new integral

$= \frac{-1}{8.5}u^{-1.7} + C$; Simplify

$= \frac{-1}{8.5(5t+4)^{1.7}} + C$; Replace *u* with $5t+4$

$= \frac{-2}{17(5t+4)^{1.7}} + C$ //

c.) $\int \frac{\tan^{-1}x}{1+x^2}\, dx$

Let $u = \tan^{-1}x$. Then $du = \frac{dx}{1+x^2}$ and $dx = (1+x^2)du$

→ $\int \frac{\tan^{-1}x}{1+x^2}\, dx = \int u du$; Substitute in *u* and *du*

$= \dfrac{u^2}{2} + C$; Replace u with $\tan^{-1} x$

$= \dfrac{(\tan^{-1} x)^2}{2} + C$ //

d.) $\displaystyle\int \frac{r^3}{\sqrt{4+r^2}}\,dr$

Let $u = r^2$ and $dv = \dfrac{r}{\sqrt{4+r^2}}\,dr$. This implies $du = 2r\,dr$ and $v = \sqrt{4+r^2}$

→ $\displaystyle\int \frac{r^3}{\sqrt{4+r^2}}\,dr = \int r^2 \frac{r}{\sqrt{4+r^2}}\,dr$; Integrate by Parts

$= r^2\sqrt{4+r^2} - \displaystyle\int 2r\sqrt{4+r^2}\,dr$; Let $u = 4 + r^2$

$= r^2\sqrt{4+r^2} - \displaystyle\int \sqrt{u}\,du$ $\quad du = 2r\,dr$

$= r^2\sqrt{4+r^2} - \dfrac{2}{3}u^{\frac{3}{2}} + C$; Replace u with $4 + r^2$

$= r^2\sqrt{4+r^2} - \dfrac{2}{3}(4+r^2)^{\frac{3}{2}} + C$ //

e.) $\displaystyle\int (\ln x)^2\,dx$

Let $u = (\ln x)^2$ and $dv = dx$. This implies $du = \dfrac{2\ln x}{x}\,dx$ and $v = x$

→ $\displaystyle\int (\ln x)^2\,dx = x(\ln x)^2 - 2\int (\ln x)\,dx$; Integrate by Parts

Let $u = \ln x$ and $dv = dx$. This implies $du = \dfrac{1}{x}\,dx$ and $v = x$.

→ $x(\ln x)^2 - 2\displaystyle\int (\ln x)\,dx = x(\ln x)^2 - 2\left(x\ln x - \int dx\right)$

$= x(\ln x)^2 - 2(x\ln x - x) + C$ //

2.) Use Newton's method to approximate the given number correct to eight decimal places.

a.) $\sqrt[7]{1000}$

→ $f(x) = x^7 - 1000 = 0$; Rewrite as $f(x)$

→ $f'(x) = 7x^6$; Find $f'(x)$

→ $x_{n+1} = x_n - \dfrac{x_n^{\,7} - 1000}{7x_n^{\,6}}$; Use Newton's Method

We need to find approximations until they agree to eight decimal places.

→ $x_1 = 3$
$x_2 \approx 2.76739173$
$x_3 \approx 2.69008741$
$x_4 \approx 2.68275645$
$x_5 \approx 2.68269580$
$x_6 \approx 2.68269580$

Thus, $\sqrt[7]{1000} \approx 2.68269580$, to eight decimal places. //

3.) The velocity function (in meters per second) is given for a particle moving along a line. Find (a.) the displacement, and (b.) the distance traveled by the particle during the given time interval

$$v(t) = t^2 - 2t - 8 \qquad 1 \le t \le 6$$

(a.) Displacement

$$= \int_1^6 (t^2 - 2t - 8)\,dt \quad ; \quad \text{Set up Integral}$$

$$= \left[\frac{1}{3}t^3 - t^2 - 8t\right]_1^6 \quad ; \quad \text{Evaluate Integral}$$

$$= (72 - 36 - 48) - \left(\frac{1}{3} - 1 - 8\right)$$

$$= -\frac{10}{3}\text{ m} \quad //$$

(b.) Distance Traveled

$$= \int_1^6 \left|(t^2 - 2t - 8)\right| dt \quad ; \quad \text{Set up Integral}$$

$$= \int_1^6 \left|(t-4)(t+2)\right| dt \quad ; \quad \text{Determine when } v(t) < 0$$

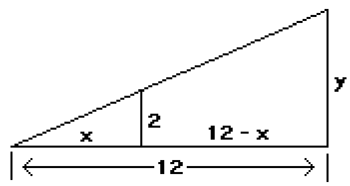
$$= \int_1^4 -(t^2 - 2t - 8)\,dt + \int_4^6 (t^2 - 2t - 8)\,dt$$

$$= \left[-\frac{1}{3}t^3 + t^2 + 8t\right]_1^4 + \left[\frac{1}{3}t^3 - t^2 - 8t\right]_4^6$$

$$= \frac{98}{3}\text{ m} \quad //$$

4.) A spotlight on the ground shines on a wall 12 m away. If a man 2 m tall walks from the spotlight toward the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing when he is 4 m from the building?

We are given that $\frac{dx}{dt} = 1.6$ m/s. By similar triangles,



$$\frac{y}{12} = \frac{x}{2} \quad \Rightarrow \quad y = \frac{24}{x} \qquad (\dagger)$$

If we differentiate (†) with respect to *t*, we get

$$\frac{dy}{dt} = -\frac{24}{x^2}\frac{dx}{dt} = -\frac{24}{x^2}(1.6) \qquad (\dagger\dagger)$$

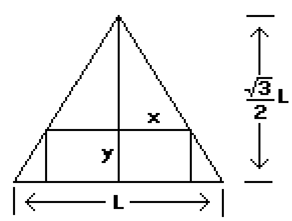
When the man is 4 m from the building, $x = 8$. Plugging this value of x into (††)

$$\frac{dy}{dt} = -\frac{24(1.6)}{8^2} = -0.6 \text{ m/s}$$

Thus, the shadow is decreasing at a rate of 0.6 m/s. //

5.) Find the dimension of the largest area that can be inscribed in an equilateral triangle of side L if one side of the rectangle lies on the base of the triangle.

The height h of the equilateral triangle with sides of length L is $\frac{\sqrt{3}}{2}L$, since



$$h^2 + \left(\frac{L}{2}\right)^2 = L^2 \quad \Rightarrow \quad h^2 = L^2 - \frac{1}{4}L^2 = \frac{3}{4}L^2$$

$$\Rightarrow \quad h = \frac{\sqrt{3}}{2}L$$

Using similar triangles,

$$\frac{\frac{\sqrt{3}}{2}L - y}{x} = \frac{\frac{\sqrt{3}}{2}L}{\frac{L}{2}} \quad \Rightarrow \quad y = \frac{\sqrt{3}}{2}(L - 2x) \qquad (\dagger)$$

The area of the inscribed rectangle is

$$A(x) = (2x)y = (2x)\left(\frac{\sqrt{3}}{2}(L - 2x)\right) = \sqrt{3}Lx - 2\sqrt{3}x^2 \qquad (\dagger\dagger)$$

where $0 \le x \le \frac{L}{2}$. Now, we can find the critical points of (††), namely

$$A'(x) = \sqrt{3}L - 4\sqrt{3}x = 0 \rightarrow x = \frac{\sqrt{3}L}{4\sqrt{3}} = \frac{L}{4}$$

Since $A(0) = A\left(\frac{L}{2}\right) = 0$, the maximum occurs when $x = \frac{L}{4}$, and

$y = \frac{\sqrt{3}}{2}L - \frac{\sqrt{3}}{4}L = \frac{\sqrt{3}}{4}L$ (via equation (†)). Thus, the dimensions of the largest inscribed rectangle are $\frac{L}{2}$ by $\frac{\sqrt{3}}{4}L$. //

6.) Find the derivative of the following function.

$$g(x) = \int_{2x}^{3x} \frac{u^2 - 1}{u^2 + 1}\, du$$

$$\begin{aligned}
\rightarrow \quad g(x) &= \int_{2x}^{3x} \frac{u^2 - 1}{u^2 + 1} \\
&= \int_{2x}^{0} \frac{u^2 - 1}{u^2 + 1} du + \int_{0}^{3x} \frac{u^2 - 1}{u^2 + 1} du \\
&= -\int_{0}^{2x} \frac{u^2 - 1}{u^2 + 1} du + \int_{0}^{3x} \frac{u^2 - 1}{u^2 + 1} du \\
\rightarrow \quad g'(x) &= -\frac{(2x)^2 - 1}{(2x)^2 + 1} \cdot \frac{d}{dx}(2x) + \frac{(3x)^2 - 1}{(3x)^2 + 1} \cdot \frac{d}{dx}(3x) \\
&= -2 \cdot \frac{4x^2 - 1}{4x^2 + 1} + 3 \cdot \frac{9x^2 - 1}{9x^2 + 1} \quad //
\end{aligned}$$

7.) Find the interval on which the curve

$$y = \int_{0}^{x} \frac{1}{1 + t + t^2}\, dt$$

is concave upward.

For the curve to be concave upward, we must have $y'' > 0$.

$$\begin{aligned}
\rightarrow \quad y &= \int_{0}^{x} \frac{1}{1 + t + t^2} dt \\
\rightarrow \quad y' &= \frac{1}{1 + x + x^2} \qquad ; \qquad \text{Find } y'
\end{aligned}$$

→ $y'' = \dfrac{-(1+2x)}{(1+x+x^2)^2}$; Find y''

For $y'' > 0$, we must have $-(1+2x) > 0$ since $(1 + x + x^2)^2 > 0$ for all x.

$$-(1+2x) = -1-2x > 0$$
$$-2x > 1$$
$$x < -\frac{1}{2}$$

Thus, the curve is concave upward on $\left(-\infty,\ -\dfrac{1}{2}\right)$. //

Final Exam: Thurs, June 26th at 6 pm in MCS B33

No Calculators Permitted