# Multiple Holomorphs of Dihedral and Quaternionic Groups 

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#### Abstract

The holomorph of a group $G$ is $\operatorname{Norm}_{B}(\lambda(G))$, the normalizer of the left regular representation $\lambda(G)$ in its group of permutations $B=\operatorname{Perm}(G)$. The multiple holomorph of $G$ is the normalizer of the holomorph in $B$. The multiple holomorph and its quotient by the holomorph encodes a great deal of information about the holomorph itself and about the group $\lambda(G)$ and its conjugates within the holomorph. We explore the multiple holomorphs of the dihedral groups $D_{n}$ and quaternionic (dicyclic) groups $Q_{n}$ for $n \geq 3$.


Key words: regular subgroup, holomorph
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## Introduction

The holomorph $\operatorname{Hol}(G)$ of a (finite) group $G$ can be presented in two ostensibly different forms. The first is as $G \rtimes \operatorname{Aut}(G)$ the semi-direct product of $G$ by its automorphism group where $(g, \alpha)(h, \beta)=(g \alpha(h), \alpha \beta)$. The
other is as $\operatorname{Norm}_{B}(\lambda(G))$ the normalizer in $B=\operatorname{Perm}(G)$ of the image of the left regular representation $\lambda: G \rightarrow \operatorname{Perm}(G)$ where $\lambda(g)(h)=g h$. As shown in elementary texts like Hall [9] for example, the latter is isomorphic to the former since $\operatorname{Norm}_{B}(\lambda(G))=\rho(G) \operatorname{Aut}(G)$ where $\rho(G)$ is the image of the right regular representation $\rho: G \rightarrow \operatorname{Perm}(G)$ where $\rho(g)(h)=h g^{-1}$ and $\operatorname{Aut}(G)$ is naturally embedded in $B=\operatorname{Perm}(G)$ as $\left\{\pi \in \operatorname{Norm}_{B}(\lambda(G)) \mid \pi(e)=e\right\}$ where $e$ is the identity element of $G$. One confirms that $\rho(G) \cap \operatorname{Aut}(G)$ is trivial, which together with the fact that $\rho(G) \triangleleft \operatorname{Norm}_{B}(\lambda(G))$, yields the isomorphism with the semi-direct product.

When $G$ is abelian, $\lambda(G)=\rho(G)$, but if $G$ is non-abelian, these are distinct subgroups of $\operatorname{Perm}(G)$. In either case one may show that $\operatorname{Hol}(G)=$ $\operatorname{Norm}_{B}(\lambda(G))=\operatorname{Norm}_{B}(\rho(G))$, which follows from the fact that $\lambda(G)=$ $\operatorname{Cent}_{B}(\rho(G))$ and $\rho(G)=\operatorname{Cent}_{B}(\lambda(G))$. This then motivates two questions. The first is that, given $G$, what subgroups $N \leq B=\operatorname{Perm}(G)$ have the property that $\operatorname{Norm}_{B}(N)=\operatorname{Hol}(G)$ ? (aside from $\rho(G)$ of course) The second is, how are these $N$ related to $\lambda(G)$ ? A way of answering both is embodied in the notion of the multiple holomorph of $G$ which (as defined by Miller in [11]) is the normalizer in $\operatorname{Perm}(G)$ of $\operatorname{Hol}(G)$ which we shall denote $\operatorname{NHol}(G)$. This group, and in particular the quotient $\operatorname{NHol}(G) / \operatorname{Hol}(G)$ gives an answer to both these questions in the case where such $N$ are regular subgroups of $B$ (as defined below) which are (necessarily) conjugates of $\lambda(G)$.

Whereas Miller considered the structure of $\operatorname{NHol}(G)$ for $G$ is abelian, we shall consider the class of dihedral groups $G=D_{n}$ (or order $2 n$ ) and quaternionic groups $Q_{n}$ (of order $4 n$ ) for each $n \geq 3$ and explicitly determine $\operatorname{NHol}(G)$ in each case, and also the regular subgroups that have the same holomorph/normalizer as $G$. In particular we shall show that

Theorem For $n \geq 3$

$$
\begin{aligned}
& \operatorname{NHol}\left(D_{n}\right) / \operatorname{Hol}\left(D_{n}\right) \cong\left\{u \in U_{n} \mid u^{2}=1\right\} \\
& \operatorname{NHol}\left(Q_{n}\right) / \operatorname{Hol}\left(Q_{n}\right) \cong\left\{u \in U_{2 n} \mid u^{2}=1\right\}
\end{aligned}
$$

where $U_{n}$ and $U_{2 n}$ are respectively the units $\bmod n$ and $2 n$.
We shall show first that $\operatorname{Hol}(N)=\operatorname{Hol}\left(D_{n}\right)$ if and only if $N$ is a conjugate of $\lambda\left(D_{n}\right)$ and $N \triangleleft \operatorname{Hol}\left(D_{n}\right)$ then find those conjugates of $\lambda\left(D_{n}\right)$ that are
normal in $\operatorname{Hol}\left(D_{n}\right)$. From this, we construct a subgroup $M_{n} \leq B$ such that $\operatorname{NHol}\left(D_{n}\right) \cong \operatorname{Hol}\left(D_{n}\right) \rtimes M_{n}$ where $M_{n} \cong\left\{u \in U_{n} \mid u^{2}=1\right\}$ where $\left|M_{n}\right|$ is precisely the number of conjugates of $\lambda\left(D_{n}\right)$ that are normal in $\operatorname{Hol}\left(D_{n}\right)$. Subsequently we shall consider the groups $Q_{n}$ and show how the multiple holomorphs of these are related to those of the dihedral group of the same order.

The question of which groups not necessarily isomorphic to $G$ have the same holomorph as $G$ has appeared in the literature. In particular, Mills [12] uses the term multiple holomorph in his study of which abelian groups $N$ have holomorphs which are isomorphic to a given abelian group $G$ where $N$ may not be isomorphic to $G$. Mills himself indicates that this is quite different than Miller's definition. In particular, conjugacy considerations do not apply at all, of course, if $N \not \approx G$. As such, the structure of $\mathrm{NHol}(G) / \operatorname{Hol}(G)$ has no direct bearing. However if there is another nonisomorphic group of order $|G|$ whose holomorph is isomorphic to that of $G$ then there is an isomorphism between the multiple holomorphs of each, a fact which we shall use in our analysis of $D_{n}$ and $Q_{n}$.

The inspiration for this work arose from the author's study of regular permutation groups, which are an essential part of the enumeration of Hopf-Galois structures on separable field extensions. The key result in this area (as given in [8]) is that such structures are in one-to-one correspondence with regular subgroups of a certain symmetric group, normalized by a fixed regular subgroup. Understanding normalizers of regular permutation groups has connections with the classical theory of the group holomorph as elucidated in [4] and [2], and, as indicated above, in the author's own recent work.

## 1 Regularity and (Multiple) Holomorphs

To start with, we restrict our attention to certain classes of subgroups of $B$, namely those which, like $\lambda(G)$ and $\rho(G)$, are regular permutation groups.

Definition 1.1: For a finite set $Z$, a subgroup $N \leq \operatorname{Perm}(Z)$ is regular if any two of the following properties holds:

1. $n(z)=z$ for any $z \in Z$ implies that $n=e_{N}$ the identity of $N$
2. $N$ acts transitively on $Z$
3. $|N|=|Z|$

The first condition is used (e.g. [15]) as the definition of a semi-regular subgroup, which will be needed in the discussion to follow. The subgroups $\lambda(G)$ and $\rho(G)$ are canonically regular by the way they are defined. However regularity is not tied to just these representations. In fact, we have the following (in the author's opinion) very important fact about conjugacy classes of regular subgroups, which is explicitly proven in [5] (but also used in Ch. II, Section 19 of [6]) which we paraphrase here

Proposition 1.2: Any two regular subgroups of $B=\operatorname{Perm}(Z)$ which are isomorphic (as abstract groups) are, in fact, conjugate in $\operatorname{Perm}(Z)$.

An obvious (yet important) consequence of this is that normalizers are similarly related.

Corollary 1.3: If $M$ and $N$ are isomorphic regular subgroups of $B=\operatorname{Perm}(Z)$ where $M=\sigma N \sigma^{-1}$ then $\operatorname{Norm}_{B}(M)=N o r m_{B}\left(\sigma N \sigma^{-1}\right)=\sigma N o r m_{B}(N) \sigma^{-1}$.

An essential means for detecting when a regular subgroup of the normalizer of a given regular permutation group has the same normalizer is the following.

Proposition 1.4:[10, 3.8] Given a regular subgroup $N$ of $B$, if $M$ is a normal regular subgroup of $\operatorname{Norm}_{B}(N)$ then $\operatorname{Norm}_{B}(N) \leq N o r m b(M)$ and if $|\operatorname{Aut}(N)|=|\operatorname{Aut}(M)|$ then $N o r m_{B}(N)=\operatorname{Norm}_{B}(M)$.

Conversely, if $N$ is regular and $\operatorname{Norm}_{B}(N)=\operatorname{Hol}(G)$ then $N \triangleleft \operatorname{Hol}(G)$ so if $N \cong \lambda(G)$ then $N$ is a normal subgroup of $\operatorname{Hol}(G)$ conjugate to $\lambda(G)$ in $B$. Another important consequence of Proposition 1.2 is that $N$, as a regular subgroup of $\operatorname{Perm}(Z)$, is no different from $\lambda(N) \leq \operatorname{Perm}(N)$ with respect to the construction of $\operatorname{Norm}_{\operatorname{Perm}(Z)}(N)$ as compared to $\operatorname{Hol}(N)=$ $\operatorname{Norm}_{\operatorname{Perm}(N)}(\lambda(N))$. Specifically, one may show the following:

Proposition 1.5:[10, 3.6,3.7] If $N$ is a regular subgroup of $B=\operatorname{Perm}(Z)$, then

$$
\operatorname{Norm}_{B}(N)=\operatorname{Cent}_{B}(N) A_{(N, z)}
$$

where $A_{(N, z)}=\left\{\pi \in \operatorname{Norm}_{B}(N) \mid \pi(z)=z\right\}$ for any chosen $z \in Z$. That is $\operatorname{Norm}_{B}(N) \cong \operatorname{Hol}(N)=\operatorname{Norm}_{\operatorname{Perm}(N)}(\lambda(N))=\rho(N) \operatorname{Aut}(N)$. Moreover, for any $z_{1}, z_{2} \in Z$ one has that $\operatorname{Cent}_{B}(N) A_{\left(N, z_{1}\right)}=\operatorname{Cent}_{B}(N) A_{\left(N, z_{2}\right)}$.

The last statement above is particularly a consequence of the fact that any two $A_{\left(N, z_{1}\right)}$ and $A_{\left(N, z_{2}\right)}$ are conjugate by any element of $\pi \in \operatorname{Norm}_{B}(N)$ such that $\pi\left(z_{1}\right)=z_{2}$.

The appearance of $\operatorname{Cent}_{B}(N)$ bears some discussion, especially in the context of regularity. Following [8] we have the following:

Definition 1.6: For $N$ a regular subgroup of $\operatorname{Perm}(Z)$ define the opposite group to $N$ as $N^{o p p}=\operatorname{Cent}_{B}(N)$.

We include here several easily verified properties of the opposite group which can be found in section 3 of [10] for example.

Lemma 1.7: For $N$ a regular subgroup of $B=\operatorname{Perm}(Z)$

1. $N \cap N^{o p p}=Z(N)$ the center of $N$
2. $N^{\text {opp }}$ is also a regular subgroup of $B$
3. $N=N^{\text {opp }}$ if and only if $N$ is abelian
4. $\left(N^{o p p}\right)^{o p p}=N$
5. $\operatorname{Norm}_{B}(N)=\operatorname{Norm}_{B}\left(N^{o p p}\right)$

Proof. Statement (1) is trivial and immediately implies (3). Statement (2) is [8, Lemma 2.4.2] where $N^{\text {opp }}$ may be constructed explicitly. Specifically, if we select a distinguished element called '1' in $Z$ then $N^{o p p}=\left\{\phi_{n} \mid n \in N\right\}$ where for $z \in Z$ one has $\phi_{n}(z)=n_{z} n(1)$ where $n_{z} \in N$ is that (unique!) element such that $n_{z}(1)=z$. Statement (4) comes from observing that
$N \leq\left(N^{o p p}\right)^{o p p}$ and since $N^{o p p}$ is regular then so must ( $\left.N^{o p p}\right)^{o p p}$ be regular. As such, $N \leq\left(N^{o p p}\right)^{o p p}$ implies $|N| \leq\left|\left(N^{o p p}\right)^{o p p}\right|$ where now $\left|\left(N^{o p p}\right)^{o p p}\right|=$ $|N|$ by regularity. Statement (5) follows from Proposition 1.4 since certainly $N^{o p p} \triangleleft \operatorname{Norm}_{B}(N)$ and $N \cong N^{o p p}$.

The last statement is an analogue of the equality $\lambda(G) A u t(G)=\rho(G) A u t(G)$ for $B=\operatorname{Perm}(G)$ and the relationship of $\rho(G)$ to $\lambda(G)$ broadly generalized is encoded by the multiple holomorph.

Definition 1.8: For $G$ a (finite) group with $B=\operatorname{Perm}(G)$ and $\operatorname{Hol}(G)=$ $\operatorname{Norm}_{B}(\lambda(G))$ define

$$
\begin{aligned}
\mathcal{H}(G) & =\left\{N \leq \operatorname{Hol}(G) \mid N \text { is regular and } N o r m_{B}(N)=\operatorname{Hol}(G)\right\} \\
& =\{N \triangleleft \operatorname{Hol}(G) \mid N \text { is regular and } N \cong G\} \\
& =\left\{N \triangleleft \operatorname{Hol}(G) \mid N=\sigma \lambda(G) \sigma^{-1} \text { for some } \sigma \in B\right\}
\end{aligned}
$$

The above, along with Corollary 1.3 are the ingredients for Miller's characterization of the multiple holomorph. Recall that the multiple holomorph is the normalizer $\operatorname{NHol}(G)=\operatorname{Norm}_{B}(\operatorname{Hol}(G))$ where $B=\operatorname{Perm}(G)$. We have the following which is essentially the first two paragraphs of [11].

Theorem 1.9: The elements of $\mathrm{NHol}(G) / \operatorname{Hol}(G)$ determine those regular subgroups of $N \leq \operatorname{Perm}(G)$ such that $\operatorname{Norm}_{B}(N)=\operatorname{Norm}_{B}(\lambda(G))$ and which, in turn, are precisely the conjugates of $\lambda(G)$ that are normal subgroups of $\operatorname{Hol}(G)$. That is $\mathrm{NHol}(G) / \operatorname{Hol}(G)$ acts transitively and fixed-point-freely on $\mathcal{H}(G)$.

Proof. If $\sigma \in \operatorname{NHol}(G)$ then $\sigma \operatorname{Norm}_{B}(\lambda(G)) \sigma^{-1}=\operatorname{Norm}_{B}(\lambda(G))$ and by Corollary 1.3

$$
\operatorname{Norm}_{B}\left(\sigma \lambda(G) \sigma^{-1}\right)=\operatorname{Norm}_{B}(\lambda(G))
$$

and so $\sigma \lambda(G) \sigma^{-1}$ is a regular subgroup of $B$ with the same holomorph as $G$. Moreover $\sigma \lambda(G) \sigma^{-1} \triangleleft \operatorname{Norm}_{B}(\lambda(G))$ since $\sigma \lambda(G) \sigma^{-1} \triangleleft N o r m_{B}\left(\sigma \lambda(G) \sigma^{-1}\right)=$ $\operatorname{Norm}_{B}(\lambda(G))$. Conversely, if for some $\sigma \in B$ one has $\sigma \lambda(G) \sigma^{-1} \triangleleft \operatorname{Norm}_{B}(\lambda(G))$
then by Proposition 1.4 clearly $\operatorname{Norm}_{B}\left(\sigma \lambda(G) \sigma^{-1}\right)=\operatorname{Norm}_{B}(\lambda(G))$ and therefore $\sigma \in \operatorname{NHol}(G)$. It should also be noted that if $N=\sigma \lambda(G) \sigma^{-1}$ then $\tau \lambda(G) \tau^{-1}=N$ if and only if $\tau \in \sigma \operatorname{Hol}(G)$, the coset represented by $\sigma$. Conversely, if $\sigma \lambda(G) \sigma^{-1}=\tau \lambda(G) \tau^{-1}$ then $\tau \sigma^{-1} \in \operatorname{Hol}(G)$. As such the distinct cosets in $\operatorname{NHol}(G) / \operatorname{Hol}(G)$ parametrize the normal subgroups of $\operatorname{Hol}(G)$ that are conjugate to $\lambda(G)$.

In [11] and [6] this construction for non-abelian groups is mentioned in passing by considering the natural case of $\lambda(G) \neq \rho(G)$ which have the same holomorph. It is observed that if one picks $\tau \in B$ such that $\tau \lambda(G) \tau^{-1}=\rho(G)$ (which must exist by Proposition 1.2) then adjoining $\tau$ to $\operatorname{Hol}(G)$ yields a subgroup of $\operatorname{NHol}(G)$ called the double holomorph.

Of course $\operatorname{NHol}(G)$ may be larger, which is precisely the cases we are interested in here. However, it is more efficacious to work with the quotient $T(G)=\operatorname{NHol}(G) / \operatorname{Hol}(G)$ and how it acts as a regular permutation group on $\mathcal{H}(G)$. We explore some of the properties of $T(G)$ in particular the role of the mapping $N \mapsto N^{o p p}$ for elements of $\mathcal{H}(G)$. As indicated above, $T(G)$ acts regularly on $\mathcal{H}(G)$. If for notational simplicity we identify $\lambda(G)$ with $G$ then each $N \in \mathcal{H}(G)$ is uniquely of the form $G_{\alpha}=\alpha G \alpha^{-1}$, for each $\alpha \in T(G)$. The regularity will allow us to show some interesting properties of $T(G)$ that are independent of the structural properties of $G$, except for whether $G$ (and hence every $N \in \mathcal{H}(G)$ ) is abelian or not. We start by observing that $G=G_{1}$ and that for $\alpha, \beta \in T(G)$ that $\alpha G_{\beta} \alpha^{-1}=G_{\alpha \beta}$.

Proposition 1.10: If $N \in \mathcal{H}(G)$ where $N=G_{\alpha}$ then $N^{\text {opp }} \in \mathcal{H}(G)$. Also, there exists a unique $\delta \in T(G)$ such that $N^{o p p}=\delta N \delta^{-1}=G_{\delta \alpha}$. Moreover, if $G$ is abelian then $\delta=1$ and if $G$ is non-abelian then $|\delta|=2$.

Proof. The first sentence is a consequence of Theorem 1.9, Lemma 1.7 and the definition of $T(G)$ and its acting regularly on $\mathcal{H}(G)$. The second is due to the fact that the conjugate of a centralizer is the centralizer of the conjugate, to wit $\delta N^{o p p} \delta^{-1}=\left(\delta N \delta^{-1}\right)^{o p p}$. If $G$ is abelian then $N^{o p p}=N$ so that $\delta=1$ automatically by regularity. If $G$ is non-abelian then so is any such $N$ and so $N^{o p p} \neq N$ which means that $G_{\delta \alpha} \neq G_{\alpha}$ and so $\delta \neq 1$ by regularity. Furthermore $\delta N^{o p p} \delta^{-1}=\left(\delta N \delta^{-1}\right)^{o p p}=\left(N^{o p p}\right)^{\text {opp }}=N$ but this means that $G_{\delta^{2} \alpha}=G_{\alpha}$ which implies $\delta^{2}=1$, again by regularity!

Corollary 1.11: If $G$ is non-abelian then $|T(G)|=|\mathcal{H}(G)|$ is even.
For $G=D_{n}$ as we will see it goes even further, $T(G)$ is an elementary abelian 2-group. The next result is a somewhat more abstract fact about how the ( ) opp operation is realized by elements of $T(G)$.

Proposition 1.12: Given $G=G_{1} \in \mathcal{H}(G)$, if $\tau \in T(G)$ is that element such that $G^{\text {opp }}=G_{\tau}=\tau G \tau^{-1}$ then if $\alpha \in T(G)$ is arbitrary and $\delta \in T(G)$ is such that $\delta G_{\alpha} \delta^{-1}=G_{\alpha}^{\text {opp }}$ and if $\beta \in T(G)$ is such that $\beta G^{o p p} \beta^{-1}=G_{\beta}^{o p p}$ then $\alpha=\beta$ and $\delta \alpha=\alpha \tau$ in $T(G)$.

Proof. The basic layout is as follows:

and where by the way $T(G)$ acts and by regularity of this action we have $G_{\alpha}^{o p p}=G_{\delta \alpha}=G_{\beta \tau}$. However, since $G_{\alpha}^{o p p}=\left(\alpha G_{1} \alpha^{-1}\right)^{\text {opp }}=\alpha G_{1}^{o p p} \alpha^{-1}=$ $\alpha G_{\tau} \alpha^{-1}=G_{\alpha \tau}$ then we have $G_{\alpha}^{o p p}=G_{\alpha \tau}$ whence $\alpha \tau=\beta \tau$ and so $\alpha=\beta$. Lastly, we have that $\delta \alpha=\alpha \tau$ where $|\delta|=2$ by Proposition 1.12.

Note, in the above setup, we cannot conclude necessarily that $\delta=\tau$ unless, of course, $T(G)$ is abelian.

Corollary 1.13: In the above setup, if $G$ is non-abelian and $T(G)$ is abelian then $\delta=\tau$ for all $\alpha \in T(G)$. That is, for any $N \in \mathcal{H}(G)$ one has $N^{\text {opp }}=$ $\tau N \tau^{-1}$ for that $\tau$ such that $G_{\tau}=G^{o p p}$.

Another somewhat interesting consequence of the above is that if $G$ and $T(G)$ are non-abelian and $T(G)$ has only one element of order 2 , then $Z(T(G))=\langle\tau\rangle$.

We should note that if one looks at the definition of the double holomorph in most sources, it is the aforementioned extension of $\operatorname{Hol}(G)$ by this element whose square lies in $\operatorname{Hol}(G)$. However in [6] the authors call the group generated by $\lambda(G)$ and this element the double holomorph. But
this is not quite the same object since it does not contain the entire holomorph of $G$. Also we should note that, in many cases, $\operatorname{Hol}(G)$ may contain many regular subgroups isomorphic (and therefore conjugate) to $\lambda(G)$ but only those normal in $\operatorname{Hol}(G)$ have the same holomorph. Miller [11] shows that for abelian groups, $T(G)$ is trivial (and therefore $\mathcal{H}(G)=\{\lambda(G)\}$ ) in many instances, such as when the order of $G$ is not divisible by 8 . For the rest, he shows that $T(G)$ may still be trivial, or at most cyclic of order 2 or the Klein four-group, depending on the factors making up the 2-power component of $G$.

## 2 Dihedral and Quaternionic Groups

We shall use the following presentation of the dihedral group of order $2 n$ and quaternionic group $Q_{n}$ of order $4 n$ for $n \geq 3$ :

$$
\begin{aligned}
D_{n} & =\left\langle x, t \mid x^{n}=t^{2}=1, x t=t x^{-1}\right\rangle \\
& =\left\{t^{a} x^{b} \mid a \in \mathbb{Z}_{2}, b \in \mathbb{Z}_{n}\right\} \\
Q_{n} & =\left\langle x, t \mid x^{2 n}=1, x^{n}=t^{2}, x t=t x^{-1}\right\rangle \\
& =\left\{t^{a} x^{b} \mid a \in \mathbb{Z}_{2}, b \in \mathbb{Z}_{2 n}\right\}
\end{aligned}
$$

It is a standard fact that $\operatorname{Aut}\left(D_{n}\right)$ is isomorphic to $\operatorname{Hol}\left(\mathbb{Z}_{n}\right) \cong \mathbb{Z}_{n} \rtimes U_{n}$, where $U_{n}=\mathbb{Z}_{n}^{*}$, specifically

$$
\begin{gathered}
A u t\left(D_{n}\right)=\left\{\phi_{(i, j)} \mid i \in \mathbb{Z}_{n}, j \in U_{n}\right\} \text { where } \\
\phi_{(i, j)}\left(t^{a} x^{b}\right)=t^{a} x^{i a+j b} \text { and } \phi_{\left(i_{2}, j_{2}\right)} \circ \phi_{\left(i_{1}, j_{1}\right)}=\phi_{\left(i_{2}+j_{2} i_{1}, j_{2} j_{1}\right)}
\end{gathered}
$$

It is also a classical fact (e.g. [6, pp.169-170]), that $\operatorname{Aut}\left(D_{2 n}\right) \cong \operatorname{Aut}\left(Q_{n}\right)$ for $n \geq 3$, specifically $\operatorname{Aut}\left(D_{2 n}\right) \cong \operatorname{Aut}\left(Q_{n}\right) \cong \operatorname{Hol}\left(C_{2 n}\right)$. In fact, still more is true (and this too is classical) namely that $\operatorname{Hol}\left(D_{2 n}\right) \cong \operatorname{Hol}\left(Q_{n}\right)$. We can go one step further and observe that, since the underlying sets $D_{2 n}$ and $Q_{n}$ are identical, namely $Z=\left\{t^{a} x^{b} \mid a=0,1 ; b=0, \ldots, 2 n-1\right\}$, then we have the following from [10] which we quote verbatim:

Proposition 2.1:[10, 3.10] For $n \geq 3, \operatorname{Hol}\left(D_{2 n}\right)=\operatorname{Hol}\left(Q_{n}\right)$ as subgroups of $\operatorname{Perm}(Z)$.

Proof. If $t^{a} x^{b}$ is in $Z$ then we can define $\rho_{d}\left(t^{a} x^{b}\right)(z)$ to be $z\left(t^{a} x^{b}\right)^{-1}$ where the product is viewed with respect to the group law in $D_{2 n}$ and $\rho_{q}\left(t^{a} x^{b}\right)(z)=$ $z\left(t^{a} x^{b}\right)^{-1}$ viewed with respect to the group law in $Q_{n}$. If we define $\mathcal{A}=$ $\left\{\phi_{i, j}\right\}$ to be the subgroup of $\operatorname{Perm}(Z)$ given by $\phi_{i, j}\left(t^{c} x^{d}\right)=t^{c} x^{i c+j d}$, then $\mathcal{A}$ is seen to be the automorphism group of both $D_{2 n}$ and $Q_{n}$ simultaneously. One can then verify that:
(a) $\rho_{q}\left(x^{b}\right) \phi_{i, j}=\rho_{d}\left(x^{b}\right) \phi_{i, j}$
(b) $\rho_{q}\left(t x^{b}\right) \phi_{i, j}=\rho_{d}\left(t x^{b+n}\right) \phi_{i+n, j}$
as permutations of $Z$, keeping in mind that in the dihedral groups $t^{-1}=$ $t$ while in the quaternionic groups $t^{2}=x^{n}$ and $t^{-1}=t^{3}=t x^{n}$, and that $n \equiv-n(\bmod 2 n)$. The point is that $\operatorname{Hol}\left(D_{2 n}\right)=\left\{\rho_{d}\left(t^{a} x^{b}\right) \phi_{i, j}\right\}=$ $\left\{\rho_{q}\left(t^{a} x^{b}\right) \phi_{i, j}\right\}=\operatorname{Hol}\left(Q_{n}\right)$ and that the image of the right regular representation of $D_{2 n}$, (respectively $Q_{n}$ ) is a normal subgroup of $\operatorname{Hol}\left(Q_{n}\right)$ (respectively $\left.\operatorname{Hol}\left(D_{2 n}\right)\right)$ and the result follows by Proposition 1.4.

The immediate consequence of this (and Theorem 1.9) is
Corollary 2.2: $\operatorname{NHol}\left(D_{2 n}\right)=\operatorname{NHol}\left(Q_{n}\right)$ as subgroups of $\operatorname{Perm}(Z)$ and $\left|\mathcal{H}\left(D_{2 n}\right)=\left|\mathcal{H}\left(Q_{n}\right)\right|\right.$ for $n \geq 3$.

As to the actual computation of $\operatorname{NHol}\left(D_{n}\right)$ and $\mathcal{H}\left(D_{n}\right)$, some initial setup is needed. We observe, and this will be important later, that $C=\langle x\rangle$ is a characteristic subgroup of $D_{n}$. Now, as $\operatorname{Hol}\left(D_{n}\right)=\operatorname{Norm}_{B}\left(\lambda\left(D_{n}\right)\right)=$ $\rho\left(D_{n}\right) \operatorname{Aut}\left(D_{n}\right)$ where $B=\operatorname{Perm}\left(D_{n}\right)$ we need to compute the left and right regular representations of $D_{n}$ as well as of one of the principal generators of $A u t\left(D_{n}\right)$ in $B$.

Lemma 2.3: These are the representations of the generators of $\lambda\left(D_{n}\right), \rho\left(D_{n}\right)$
and one order $n$ generator of $\operatorname{Aut}\left(D_{n}\right)$ as elements of $\operatorname{Perm}\left(D_{n}\right)$.

$$
\begin{aligned}
\lambda(x) & =\left(1, x, \ldots, x^{n-1}\right)\left(t, t x^{n-1}, t x^{n-2}, \ldots, t x\right) \\
\lambda(t) & =(1, t)(x, t x) \cdots\left(x^{n-1}, t x^{n-1}\right) \\
\rho(x) & =\left(1, x^{n-1}, x^{n-2}, \ldots, x\right)\left(t, t x^{n-1}, \ldots, t x\right) \\
\rho(t) & =(1, t)\left(x, t x^{n-1}\right)\left(x^{2}, t x^{n-2}\right) \cdots\left(x^{n-1}, t x\right) \\
\phi_{(1,1)} & =\left(t, t x, t x^{2}, \ldots, t x^{n-1}\right)
\end{aligned}
$$

Proof. These are based on the presentations of $D_{n}$ and $\operatorname{Aut}\left(D_{n}\right)$ above.

Although one can compute the representation of $\phi_{(0, j)}$ for $j \in U_{n}$ it turns out to not be necessary for the computations we shall be doing. We shall need, however, several operational details about the elements of $\operatorname{Hol}\left(D_{n}\right)$, such as the orders of elements, as well as determining which act without fixed-points. In particular we need to use both formulations of $\operatorname{Hol}\left(D_{n}\right)$, as $\rho\left(D_{n}\right) \operatorname{Aut}\left(D_{n}\right)$ and as the semi-direct product $D_{n} \rtimes \operatorname{Aut}\left(D_{n}\right)$ consisting of ordered pairs $\left(t^{a} x^{b}, \phi_{(i, j)}\right)$. To start with, we adopt the following convention:

$$
\begin{align*}
\left(t^{a} x^{b}, \phi_{(i, j)}\right)\left(t^{c} x^{d}\right) & =\rho\left(t^{a} x^{b}\right) \phi_{(i, j)}\left(t^{c} x^{d}\right) \\
& = \begin{cases}t^{c} x^{i c+j d-b} & a=0 \\
t^{c+1} x^{b-i c-j d} & a=1\end{cases} \tag{1}
\end{align*}
$$

One of the immediate implications of this is the following.
Lemma 2.4: The element $\left(t x^{b}, \phi_{(i, j)}\right)$ has no fixed-points for any $b, i, j$ and $\left(x^{b}, \phi_{(i, j)}\right)$ acts without fixed-points provided $i c+j d-b \not \equiv d$ for all $(c, d) \in$ $\mathbb{Z}_{2} \times \mathbb{Z}_{n}$.

Note that the identity element of $\operatorname{Hol}\left(D_{n}\right)$ is $\left(t^{0} x^{0}, \phi_{(0,1)}\right)$ which of course has $2 n$ fixed-points.

The computations we do here will require extensive analysis of powers of elements in $\operatorname{Hol}\left(D_{n}\right)$. As such we have the following.

Lemma 2.5: The powers of $\left(x^{b}, \phi_{(i, j)}\right)$ and $\left(t x^{b}, \phi_{(i, j)}\right)$ are given as follows.

$$
\begin{aligned}
\left(x^{b}, \phi_{(i, j)}\right)^{e} & =\left(x^{b g_{e}(j)}, \phi_{\left(i g_{e}(j), j^{e}\right)}\right) \\
\left(t x^{b}, \phi_{(i, j)}\right)^{2 r} & =\left(x^{(i+(j-1) b) g_{r}\left(j^{2}\right)}, \phi_{\left(i(1+j) g_{r}\left(j^{2}\right), j^{2 r}\right)}\right) \\
\left(t x^{b}, \phi_{(i, j)}\right)^{2 r+1} & =\left(t x^{\left(b+j(i+(j-1) b) g_{r}\left(j^{2}\right)\right)}, \phi_{\left(i+j i(1+j) g_{r}\left(j^{2}\right), j^{2 r+1}\right)}\right)
\end{aligned}
$$

where $g_{p}(j)=1+j+\cdots+j^{p-1}$.

Proof. All of these start with the group law in $\operatorname{Hol}\left(D_{n}\right)$, namely that

$$
\left(t^{a_{1}} x^{b_{1}}, \phi_{\left(i_{1}, j_{1}\right)}\right)\left(t^{a_{2}} x^{b_{2}}, \phi_{\left(i_{2}, j_{2}\right)}\right)=\left(t^{a_{1}} x^{b_{1}} t^{a_{2}} x^{i_{1} a_{2}+j_{1} b_{2}}, \phi_{\left(i_{1}+j_{1} i_{2}, j_{1} j_{2}\right)}\right)
$$

together with a basic induction argument. The second and third cases above give the even and odd powers of $\left(t x^{b}, \phi_{(i, j)}\right)$ which highlights a simple yet important fact, namely that $\left(t x^{b}, \phi_{(i, j)}\right)$ never has order $n$ (in $\operatorname{Hol}\left(D_{n}\right)$ ) if $n$ is odd.

Determining which regular subgroups of $\operatorname{Hol}\left(D_{n}\right)$, isomorphic to $D_{n}$, are normal in $\operatorname{Hol}\left(D_{n}\right)$ is vastly simplified by the following fact about the cyclic subgroup of order $n$ that $\lambda\left(D_{n}\right)$ and therefore any conjugate thereof contains.

Proposition 2.6: If $C_{n}=\langle x\rangle$ then $\operatorname{Norm}_{B}\left(\lambda\left(C_{n}\right)\right)=\operatorname{Norm}_{B}\left(\lambda\left(D_{n}\right)\right)$.

Proof. The group $\lambda\left(C_{n}\right)$ is a cyclic semi-regular subgroup and as such, the structure of the normalizer is known. As mentioned in passing in [14, p.334] it is an example of a twisted wreath product. By a variation of the argument in, for example $[10,5.6]$, it is isomorphic to the following semidirect product

$$
\left(\mathbb{Z}_{n} \times \mathbb{Z}_{n}\right) \rtimes\left(U_{n} \times S_{2}\right)
$$

where (as a subgroup of $B$ ) $\mathbb{Z}_{n} \times \mathbb{Z}_{n}$ corresponds to the pair of $n$-cycles that make up $\lambda(x)$, the action of $U_{n}=\mathbb{Z}_{n}^{*}$ is by multiplication of a given unit $u$ on both components of $\mathbb{Z}_{n} \times \mathbb{Z}_{n}$ simultaneously (and in $B$ naturally as a subgroup of $\operatorname{Aut}\left(D_{n}\right)$ which acts this way on $\lambda(x)$ ), and where the $S_{2}=\langle\tau\rangle$
acts coordinate-wise. That is, if $\hat{v}=(a, b) \in \mathbb{Z}_{n} \times \mathbb{Z}_{n}$ then $\tau(\hat{v})=(b, a)$ and the multiplication in general is therefore

$$
\left(\hat{v}_{1}, u_{1}, \tau^{m_{1}}\right)\left(\hat{v}_{2}, u_{2}, \tau^{m_{2}}\right)=\left(\hat{v}_{1}+u_{1} \tau^{m_{1}}\left(\hat{v}_{2}\right), u_{1} u_{2}, \tau^{m_{1}+m_{2}}\right)
$$

The wreath product structure can be more easily seen by observing that restricting to those tuples $\left(\hat{v}, u, \tau^{m}\right)$ where $u=1$ one has the centralizer $\operatorname{Cent}_{B}\left(\lambda\left(C_{n}\right)\right) \cong \mathbb{Z}_{n}$ 2 $S_{2}$ which is a classical result, appearing in [1] for example. The normalizer gives rise to the extra 'twist' by the action of $U_{n}$. In either case, the sets $X=\left\{1, x, \ldots, x^{n-1}\right\}$ and $Y=\left\{t, t x^{n-1}, \ldots, t x\right\}$ which are the supports of the cycles that comprise $\lambda(x)$, are blocks under the action of both groups. The proof is finished by observing that $\left|\left(\mathbb{Z}_{n} \times \mathbb{Z}_{n}\right) \rtimes\left(U_{n} \times S_{2}\right)\right|=2 n^{2} \phi(n)=\left|\operatorname{Hol}\left(D_{n}\right)\right|$ and so the containment $\operatorname{Norm}_{B}\left(\lambda\left(D_{n}\right)\right) \leq \operatorname{Norm}_{B}\left(\lambda\left(C_{n}\right)\right)$ must be an equality.

Corollary 2.7: If $\sigma \in \operatorname{NHol}\left(D_{n}\right)$ then $\sigma \lambda\left(D_{n}\right) \sigma^{-1} \triangleleft \operatorname{Hol}\left(D_{n}\right)$ if and only $\sigma \lambda\left(C_{n}\right) \sigma^{-1} \triangleleft \operatorname{Hol}\left(D_{n}\right)$.

As such, one only need find all distinct $\sigma\langle\lambda(x)\rangle \sigma^{-1}$ that are normal subgroups of $\operatorname{Hol}\left(D_{n}\right)$ and each will give rise to $\sigma \lambda\left(D_{n}\right) \sigma^{-1}$ in $\mathcal{H}\left(D_{n}\right)$. If we define $\Upsilon_{n}=\left\{u \in U_{n} \mid u^{2}=1\right\}$ then we will show the following.

Theorem $\mathcal{H}\left(D_{n}\right)=\left\{\left\langle\left(x, \phi_{(u+1,1)}\right),\left(t, \phi_{(0,-u)}\right)\right\rangle \mid u \in \Upsilon_{n}\right\}$ and there exists $M_{n} \leq B$ such that $M_{n} \cong \Upsilon_{n}$ and $\operatorname{Orb}_{M_{n}}\left(\lambda\left(D_{n}\right)\right)=\mathcal{H}\left(D_{n}\right)$ and consequently that $\operatorname{NHol}\left(D_{n}\right) \cong \operatorname{Hol}\left(D_{n}\right) \rtimes M_{n}$.

To show this we shall need to show that the only order $n$ semi-regular cyclic normal subgroups of $\operatorname{Hol}\left(D_{n}\right)$ are those of the form $\left\langle\left(x, \phi_{(u+1,1)}\right)\right\rangle$. To arrive at this enumeration, we need to filter the elements of $\operatorname{Hol}\left(D_{n}\right)$ by order and semi-regularity. In particular, it is not enough for an individual element to act fixed-point-freely, but for every non-trivial power of it to act fixed-point-freely as well. Once this filtration is done, determining the relevant order $n$ normal semi-regular subgroup and constructing the group $M_{n}$ is actually very straightforward.

Proposition 2.8: No element of the form $\left(t x^{b}, \phi_{(i, j)}\right)$ of order $n$ generates a normal subgroup of $\operatorname{Hol}\left(D_{n}\right)$.

Proof. We first recall the comment following Lemma 2.5 namely that if $n$ is odd then $\left(t x^{b}, \phi_{(i, j)}\right)$ never has order $n$. So we are left with the case of $n$ being even. For $n$ even, it is important to note that every element of $U_{n}$ is an odd number. As such if $\left\langle\left(t x^{b}, \phi_{(i, j)}\right)\right\rangle$ is normalized by $\operatorname{Hol}\left(D_{n}\right)$ then every generator of $\operatorname{Hol}\left(D_{n}\right)$ must conjugate $\left(t x^{b}, \phi_{(i, j)}\right)$ to an odd power. Conjugation by $\left(1, \phi_{(i, j)}\right)$ yields $\left(t x^{b}, \phi_{(i, j)}\right)^{v}$ for $v=2 r+1 \in U_{n}$ and so

$$
\begin{aligned}
\left(1, \phi_{(1,1)}\right)\left(t x^{b}, \phi_{(i, j)}\right)\left(1, \phi_{(-1,1)}\right) & =\left(t x^{1+b}, \phi_{(1+i-j, j)}\right) \\
& =\left(t x^{b}, \phi_{(i, j)}\right)^{2 r+1} \\
& =\left(t x^{b+j(i+(j-1) b) g_{r}\left(j^{2}\right)}, \phi_{\left(i+j i(1+j) g_{r}\left(j^{2}\right), j^{2 r+1}\right)}\right)
\end{aligned}
$$

which yields two important equalities that must be satisfied:

$$
\begin{aligned}
1 & =j(i+(j-1) b) g_{r}\left(j^{2}\right) \\
j^{2 r} & =1
\end{aligned}
$$

From these we have that $(i+(j-1) b)$ and $g_{r}\left(j^{2}\right)$ are in $U_{n}$. As such, since $\left(1-j^{2}\right) g_{r}\left(j^{2}\right) \equiv 0$ then $j^{2}=1$ and therefore $g_{r}\left(j^{2}\right)=r$ which means that $r$ is a unit $\bmod n$. Now, if $\left\langle\left(t x^{b}, \phi_{(i, j)}\right)\right\rangle$ is normalized by $\left(1, \phi_{(1,1)}\right)$ then since $n$ is even then $\left\langle\left(t x^{b}, \phi_{(i, j)}\right)^{2}\right\rangle$ is characteristic in $\left\langle\left(t x^{b}, \phi_{(i, j)}\right)\right\rangle$ and so must also be normalized by $\left(1, \phi_{(1,1)}\right)$. However, since $j^{2}=1,\left(t x^{b}, \phi_{(i, j)}\right)^{2}=$ $\left(x^{i+(j-1) b}, \phi_{(i(j+1), 1)}\right)$ and

$$
\left(1, \phi_{(1,1)}\right)\left(x^{i+(j-1) b}, \phi_{(i(j+1), 1)}\right)\left(1, \phi_{(-1,1)}\right)=\left(x^{i+(j-1) b}, \phi_{(i(j+1), 1)}\right)
$$

but $\left(x^{i+(j-1) b}, \phi_{(i(j+1), 1)}\right)$ must equal $\left(t x^{b}, \phi_{(i, j)}\right)^{2 v}$. But since $|h|=n$ and we have $g h g^{-1}=h^{v}$ and also $g h^{2} g^{-1}=h^{2}$ then it must be that $2 v \equiv 2(\bmod n)$. Here however $v=2 r+1$ where $r \in U_{n}$ and we have $4 r+2 \equiv 2$ and therefore $4 r \equiv 0$ which implies that $4 \equiv 0 \bmod n$. So immediately we conclude that for $n>4,\left(t x^{b}, \phi_{(i, j)}\right)$ doesn't generate a normal subgroup of $\operatorname{Hol}\left(D_{n}\right)$ of order $n$. And for $\operatorname{Hol}\left(D_{4}\right)$ one may verify that the only elements of the form $\left(t x^{b}, \phi_{(i, j)}\right)$ of order 4 are $\left(t x^{b}, \phi_{(2,1)}\right)$ for $b=0,1,2,3$, as well as $\left(t, \phi_{(2,-1)}\right),\left(t x, \phi_{(0,-1)}\right),\left(t x^{2}, \phi_{(2,1)}\right)$ and $\left(t x^{3}, \phi_{(0,-1)}\right)$, and that none of these generate a subgroup that is normalized by $\left(1, \phi_{(1,1)}\right)$.

As a result, the only possible order $n$ semi-regular normal subgroups must be generated by elements of the form $\left(x^{b}, \phi_{(i, j)}\right)$. It is interesting to
note that we did not have to check whether $\left(t x^{b}, \phi_{(i, j)}\right)$ generated a semiregular subgroups (or if it was even fixed-point-free itself at all!). Nor did we have to worry about conditions which would imply that $\left(t x^{b}, \phi_{(i, j)}\right)$ has order $n$ which would, admittedly, be rather complicated.

Proposition 2.9: The only order $n$ cyclic semi-regular normal subgroups of $\operatorname{Hol}\left(D_{n}\right)$ are of the form $\left\langle\left(x, \phi_{(u+1,1)}\right)\right\rangle$ where $u \in \Upsilon_{n}$ and each such $u$ determines a distinct subgroup.

Proof. To prove the assertion we shall need to determine which such elements (that generate semi-regular subgroups) are normalized by $(x, I)$, $(t, I),\left(1, \phi_{(1,1)}\right)$ as well as $\left(1, \phi_{(0, w)}\right)$ for $w \in U_{n}$. Observe that

$$
\begin{aligned}
\left(1, \phi_{(1,1)}\right)\left(x^{b}, \phi_{(i, j)}\right)\left(1, \phi_{(-1,1)}\right) & =\left(x^{b}, \phi_{(1+i-j, j)}\right) \\
& =\left(x^{b}, \phi_{(i, j)}\right)^{m} \text { for } m \in U_{n} \\
& =\left(x^{b g_{m}(j)}, \phi_{\left(i g_{m}(j), j^{m}\right)}\right)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
b & =b g_{m}(j) \\
1+i-j & =i g_{m}(j) \\
j^{m} & =j
\end{aligned}
$$

So $j^{m-1}=1$ which implies that $g_{m}(j)=g_{m-1}(j)+1$ and therefore that $b g_{m-1}(j)=0$ and $i g_{m-1}(j)=1-j$ which means that $\left(x^{b}, \phi_{(i, j)}\right)^{m-1}=$ $\left(x^{0}, \phi_{(1-j, 1)}\right)$ which is a non-trivial element with fixed-points unless $j=1$. We immediately observe that $(x, I)$ centralizes any element of the form $\left(x^{b}, \phi_{(i, 1)}\right)$.

Next, consider the action of $(t, I)$ which, having order 2 , must conjugate $\left(x^{b}, \phi_{(i, j)}\right)$ to a power $u$ such that $u^{2}=1$, i.e. $u \in \Upsilon_{n}$. We have

$$
\begin{aligned}
(t, I)\left(x^{b}, \phi_{(i, 1)}\right)(t, I) & =\left(x^{i-b}, \phi_{(i, 1)}\right) \\
& =\left(x^{b}, \phi_{(i, 1)}\right)^{u} \text { for } u \in \Upsilon_{n} \\
& =\left(x^{b u}, \phi_{(i u, 1)}\right)
\end{aligned}
$$

which implies that $b=b u$ and $i=u i$ and that (since $u^{2}=1$ ) that $i=$ $(u+1) b$. The requirement that $\left(x^{b}, \phi_{(i, 1)}\right)$ be fixed-point-free requires (by Lemma 2.4) $b \neq 0$ and $i \neq b$. And subsequently, any non-trivial power acts fixed-point-freely requires only that $b \in U_{n}$ and $i-b \in U_{n}$ which is consistent with the condition $i=(u+1) b$.

We now observe that

$$
\begin{aligned}
\left(1, \phi_{(0, w)}\right)\left(x^{b}, \phi_{((u+1) b, 1)}\right)\left(1, \phi_{\left(0, w^{-1}\right)}\right) & =\left(x^{w b}, \phi_{(w(u+1) b, 1)}\right) \\
& =\left(x^{b}, \phi_{((u+1) b, 1)}\right)^{w}
\end{aligned}
$$

and so we have normalization by $\left(1, \phi_{(0, w)}\right)$. Now, observe that $\left(x^{b}, \phi_{((u+1) b, 1)}\right)$ has order $n$ since $b \in U_{n}$ and concordantly $\left(x^{b}, \phi_{((u+1) b, 1)}\right)^{b^{-1}}=\left(x^{1}, \phi_{(u+1,1)}\right)$.

The final observation to make is that by Lemma 2.5 each $u \in \Upsilon_{n}$ determines a distinct subgroup $\left\langle\left(x, \phi_{(u+1,1)}\right)\right\rangle$.

Now, there are two ways to proceed from here. We can find, for each semi-regular normal subgroup of $\operatorname{Hol}\left(D_{n}\right)$ of order $n$, an element of order 2 which together generate a regular normal subgroup isomorphic to $D_{n}$. However, there is still the matter of constructing $\operatorname{NHol}\left(D_{n}\right)$ which requires working within the ambient $B=\operatorname{Perm}\left(D_{n}\right)$ and in particular the representation of $\operatorname{Hol}\left(D_{n}\right)$ as a subgroup of $B$. What we shall do is to merge these two tasks by constructing the elements of $\mathcal{H}\left(D_{n}\right)$ using the generators of the cyclic order $n$ subgroup given above and then construct $\operatorname{NHol}\left(D_{n}\right)$ as a (split) extension of $\operatorname{Hol}\left(D_{n}\right)$ by a group $M_{n} \leq B$ (whence $T_{n} \cong M_{n}$ ) that not only conjugates $\lambda(x)$ to each of these order $n$ generators, but also $\lambda\left(D_{n}\right)$ as well. For convenience we shall actually construct $M_{n}$ as the set of those $\tau \in B$ such that $\tau \lambda(x)^{-1} \tau=\rho(x) \phi_{(u+1,1)}$ where $\rho(x) \phi_{(u+1,1)}$ corresponds to $\left(x, \phi_{(u+1,1)}\right)$ in the semi-direct product.

Theorem 2.10: If for $u \in \Upsilon_{n}$ we define $\tau_{u} \in B$ by the condition $\tau\left(x^{i}\right)=x^{i}$ and $\tau_{u}\left(t x^{i}\right)=t x^{u i}$ and let $M_{n}=\left\{\tau_{u} \mid u \in \Upsilon_{n}\right\}$ then
(a) $M_{n} \cong \Upsilon_{n}$
(b) $\operatorname{Orb}_{M_{n}}\left(\lambda(x)^{-1}\right)=\left\{\rho(x) \phi_{(u+1,1)} \mid u \in \Upsilon_{n}\right\}$

Proof. By Lemma 2.3 we have $\lambda(x)^{-1}=\left(1, x^{n-1}, \ldots, x\right)\left(t, t x, \ldots, t x^{n-1}\right)$ and $\rho(x)=\left(1, x^{n-1}, x^{n-2}, \ldots, x\right)\left(t, t x^{n-1}, \ldots, t x\right)$ and $\phi_{(1,1)}=\left(t, t x, t x^{2}, \ldots, t x^{n-1}\right)$. This means that $\phi_{(k, 1)}=\left(t, t x^{k}, t x^{2 k}, \ldots\right)$ and so

$$
\rho(x) \phi_{(k, 1)}=\left(1, x^{n-1}, \ldots, x\right)\left(t, t x^{k-1}, t x^{2 k-2}, \ldots\right)
$$

and therefore, for $k=u+1$ we have

$$
\rho(x) \phi_{(u+1,1)}=\left(1, x^{n-1}, \ldots, x\right)\left(t, t x^{u}, t x^{2 u}, \ldots\right)
$$

Observe that both $\lambda(x)^{-1}$ and $\rho(x) \phi_{(u+1,1)}$ have the cycle $\left(1, x^{n-1}, \ldots, x\right)$ in common. Moreover, it's quite clear that $\phi_{(1,1)}$ (and therefore $\phi_{(u+1,1)}$ except for $u=-1$ ) fix every element of $X=\left\{1, x, \ldots, x^{n-1}\right\}$ but no elements of $Y=\left\{t, t x^{n-1}, \ldots, t x\right\}$ since $\phi_{(1,1)}\left(t x^{b}\right)=t x^{1+b}$. We want permutations $\tau_{u} \in$ $B=\operatorname{Perm}\left(D_{n}\right)$ which conjugate $\lambda(x)^{-1}$ to $\rho(x) \phi_{(u+1,1)}$ for each $u \in \Upsilon_{n}$. If we look at the cycle structure of both, it is easy to construct such elements, namely define $\tau_{u}: Y \rightarrow Y$ (and by extension from $X \cup Y=D_{n}$ to itself) by $\tau_{u}\left(t x^{i}\right)=t x^{u i}$ for each $i \in\{0, \ldots, n-1\}$. Since $u \in \Upsilon_{n}$ it is clear that $\tau_{u}^{2}=I$ for each $u$. It should be noted that the $\tau_{u}$ may have fixed-points, namely due to the existence of $i \in \mathbb{Z}_{n}$ such that $u i \equiv i(\bmod n)$. To see this, consider the case $n=8$ and $u=5$ and observe that $2 u=2$ and $4 u=4$ so that $\tau_{5}=\left(t x, t x^{5}\right)\left(t x^{3}, t x^{7}\right)$.

If we now let $M_{n}=\left\{\tau_{u} \mid u \in \Upsilon_{n}\right\}$ then it is readily verified that $\Upsilon_{n} \cong M_{n}$ by the obvious mapping $u \mapsto \tau_{u}$ and that each is an elementary abelian 2-group. That $\operatorname{Orb}_{M_{n}}\left(\lambda(x)^{-1}\right)=\left\{\rho(x) \phi_{(u+1,1)} \mid u \in \Upsilon_{n}\right\}$ is by construction. Observe, by the way, that $\tau_{-1}\left(\lambda(x)^{-1}\right)=\rho(x) \phi_{(0,1)}=\rho(x)$ which implies that $\tau_{-1}$ is the unique element of order 2 conjugating every $N$ in $\mathcal{H}$ to its opposite, as in Corollary 1.13.

Theorem 2.11: $\mathcal{H}\left(D_{n}\right)=\left\{\left\langle\left(x, \phi_{(u+1,1)}\right),\left(t, \phi_{(0,-u)}\right)\right\rangle \mid u \in \Upsilon_{n}\right\}=\operatorname{Orb}_{M_{n}}\left(\lambda\left(D_{n}\right)\right)$

Proof. We first observe that

$$
\begin{aligned}
\left(t, \phi_{(0,-u)}\right)\left(x, \phi_{(u+1,1)}\right)\left(t, \phi_{(0,-u)}\right) & =\left(t x^{-u}, \phi_{(-u(u+1),-u)}\right)\left(t, \phi_{(0,-u)}\right) \\
& =\left(t x^{-u}, \phi_{(-(u+1),-u)}\right)\left(t, \phi_{(0,-u)}\right) \\
& =\left(t x^{-u} t x^{-(u+1)}, \phi_{(-(u+1), 1)}\right) \\
& =\left(x^{-(u+1)+u}, \phi_{(-(u+1), 1)}\right) \\
& =\left(x^{-1}, \phi_{(-(u+1), 1)}\right) \\
& =\left(x, \phi_{(u+1,1))^{-1}}\right.
\end{aligned}
$$

One may now verify that $\left\langle\left(x, \phi_{(u+1,1)}\right),\left(t, \phi_{(0,-u)}\right)\right\rangle$ is normal in $\operatorname{Hol}\left(D_{n}\right)$ by looking at the action of $(t, I),(x, I),\left(1, \phi_{(1,1)}\right)$, and $\left(1, \phi_{(0, w)}\right)$ for $w \in U_{n}$, on $\left(t, \phi_{(0,-u)}\right)$. Specifically we have

$$
\begin{aligned}
(t, I)\left(t, \phi_{(0,-u)}\right)(t, I) & =\left(t, \phi_{(0,-u)}\right) \\
(x, I)\left(t, \phi_{(0,-u)}\right)\left(x^{-1}, I\right) & =\left(t, \phi_{(0,-u)}\right)\left(x, \phi_{(u+1,1)}\right)^{u-1} \\
\left(1, \phi_{(1,1)}\right)\left(t, \phi_{(0,-u)}\right)\left(1, \phi_{(-1,1)}\right) & =\left(t, \phi_{(0,-u)}\right)\left(x, \phi_{(u+1,1)}\right)^{-u} \\
\left(1, \phi_{(0, w)}\right)\left(t, \phi_{(0,-u)}\right)\left(1, \phi_{\left(0, w^{-1}\right)}\right) & =\left(t, \phi_{(0,-u)}\right)
\end{aligned}
$$

and so each of these copies of $D_{n}$ are normal in $\operatorname{Hol}\left(D_{n}\right)$. Since $\left(t, \phi_{(0,-u)}\right)$ corresponds to $\rho(t) \phi_{(0,-u)}$ in $B$ and $\rho(t) \phi_{(0,-u)}\left(t^{a} x^{b}\right)=t^{a+1} x^{u b}$ then

$$
\rho(t) \phi_{(0,-u)}=\Pi_{b \in \mathbb{Z}_{n}}\left(x^{b}, t x^{u b}\right)
$$

and since

$$
\lambda(t)=\Pi_{b \in \mathbb{Z}_{n}}\left(x^{b}, t x^{b}\right)
$$

then it is obvious that $\tau_{u} \lambda(t) \tau_{u}^{-1}=\rho(t) \phi_{(0,-u)}$ and so each $\left\langle\left(x, \phi_{(u+1,1)}\right),\left(t, \phi_{(0,-u)}\right)\right\rangle$ is regular and therefore $\operatorname{Orb}_{M_{n}}\left(\lambda\left(D_{n}\right)\right)=\mathcal{H}\left(D_{n}\right)$ where $\left|\mathcal{H}\left(D_{n}\right)\right|=\left|M_{n}\right|$.

Corollary 2.12: $\operatorname{NHol}\left(D_{n}\right) \cong \operatorname{Hol}\left(D_{n}\right) \rtimes M_{n}$ where $M_{n}$ acts on $\operatorname{Hol}\left(D_{n}\right)$ by conjugation, that is $T\left(D_{n}\right) \cong M_{n}$. Similarly $T\left(Q_{n}\right) \cong M_{2 n}$ by virtue of the relationship between $Q_{n}$ and $D_{2 n}$. Also, since we may regard $\operatorname{Hol}\left(D_{2 n}\right)$ as equal to $\operatorname{Hol}\left(Q_{n}\right)$ then $\operatorname{Orb}_{M_{n}}\left(\lambda\left(Q_{n}\right)\right)=\mathcal{H}\left(Q_{n}\right)$.

Proof. That $M_{n}$ conjugates $\rho(x)$ to another subgroup of $\operatorname{Hol}\left(D_{n}\right)$ is immediate from Theorem 2.10. The one verification to make is that $\operatorname{Aut}\left(D_{n}\right)$ is conjugated to a subgroup of $\operatorname{Hol}\left(D_{n}\right)$. We have more since $\tau_{u} \phi_{(i, j)} \tau_{u}^{-1}=$ $\phi_{(u i, j)}$ and so we have that $\operatorname{Aut}\left(D_{n}\right)$ is actually normalized by $M_{n}$, even though, of course, $\operatorname{Aut}\left(D_{n}\right)$ is not itself a normal subgroup of $\operatorname{Hol}\left(D_{n}\right)$. This is consistent with Mills' determination of the (outer) automorphisms of holomorphs of abelian groups in [13] since $\operatorname{Aut}\left(D_{n}\right) \cong \operatorname{Hol}\left(C_{n}\right)$. Specifically he shows [13, Theorem 4] that the outer automorphism group of $\operatorname{Hol}\left(C_{n}\right)$ must be an abelian 2-group. Lastly, the conjugates of $\lambda\left(Q_{n}\right)$ under $N H o l\left(Q_{n}\right) / \operatorname{Hol}\left(Q_{n}\right)$ are those by $\operatorname{NHol}\left(D_{2 n}\right) / \operatorname{Hol}\left(D_{2 n}\right)$ since $\operatorname{NHol}\left(Q_{n}\right)=$ $\operatorname{NHol}\left(D_{2 n}\right)$ and $\operatorname{Hol}\left(Q_{n}\right)=\operatorname{Hol}\left(D_{2 n}\right)$.

## 3 Other Examples

With the case of $D_{n}$ and $Q_{n}$ in mind, conjecturally it seems possible that if $G$ is non-abelian and $T(G)$ is abelian then $T(G)$ is generally (if not always) an elementary abelian 2 -group. For example, Carnahan and Childs show in [3, Theorem 4] that when $G$ is simple there are only two regular embeddings of $G$ in its holomorph, to wit $\mathcal{H}(G)=\{\lambda(G), \rho(G)\}$. Furthermore, empirical evidence may be found using the GAP [7] computer algebra system. Specifically, for all groups up to order 24, with the exception of two of the groups of order 16 where $T(G)$ is non-abelian, one finds that $T(G)$ is an elementary abelian 2-group. From another direction, a careful reading of Miller shows that for an abelian group $A$, since $\operatorname{Hol}(A)$ is isomorphic to the direct product of the holomorphs of the Sylow subgroups of $A$, that $T(A)$ is similarly a direct product of $T\left(A_{i}\right)$ for each Sylow subgroup $A_{i}$ of $A$. If one can work out how $T$ behaves upon taking group extensions in general, then using Miller's characterization of $T(A)$ for $A$ abelian, one may perhaps readily determine $T(S)$ for any solvable group $S$.

It also seems likely that the methods employed in this development may be extended to deal with all split extensions of the form $\mathbb{Z}_{n} \rtimes \mathbb{Z}_{2}$ besides just $D_{n}$. The centrality of $\Upsilon_{n}$ in the enumeration of $T\left(D_{n}\right)$ is clear, but moreover $\Upsilon_{n}$ exactly parametrizes these split extensions. (Thanks to Mark Steinberger for reminding me of this.)

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