MA294 Lecture

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February 27, 2024

Cycle Structure and Orbits

We have seen the 'table' notation for elements of S_n, e.g.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 5 & 2 & 6 \end{pmatrix} \in S_6$$

but as n increases, this notation becomes cumbersome.

There is a more efficient notation for permutations built on the notion of a 'cycle'.

Definition

For $S_n = Perm(\{1,\ldots,n\})$ the k-cycle $(i_1, i_2, \ldots, i_k)$ is that permutation σ which acts as follows:

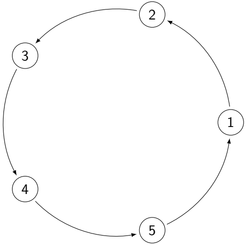
$$\begin{aligned}\sigma(i_1) &= i_2\\ \sigma(i_2) &= i_3\\ &\vdots\\ \sigma(i_{k-1}) &= i_k\\ \sigma(i_k) &= i_1\end{aligned}$$

and where $\sigma(x) = x$ for those $x \notin \{i_1, \ldots, i_k\}$.

The usage of the term 'cycle' is to highlight the fact that the permutation moves each i_t to i_{t+1} in a loop back all the way to i_1.

And again, those i not 'in the loop' are left fixed by the cycle.

For example, consider $\sigma = (1,2,3,4,5) \in S_8$



i.e. $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 4$, $\sigma(4) = 5$, $\sigma(5) = 1$, and $\sigma(6) = 6$, $\sigma(7) = 7$, $\sigma(8) = 8$

Table Notation: $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 6 & 7 & 8 \end{pmatrix}$.

Beyond individual cycles, we can look at products of cycles.

Definition

Two cycles $(i_1, i_2, \ldots, i_k)$ and $(j_1, j_2, \ldots, j_l)$ in S_n are disjoint if $\{i_1, i_2, \ldots, i_k\} \cap \{j_1, \ldots, j_l\} = \emptyset$.

e.g. $(1, 3, 5)$ and $(2, 7)$ are disjoint but $(1, 2)$ and $(2, 3, 4)$ are not.

Proposition

If $\sigma = (i_1, \ldots, i_k)$ and $\tau = (j_1, \ldots, j_l)$ are disjoint cycles, then $\sigma \circ \tau = \tau \circ \sigma$.

Why? Basically

$$(i_1, \ldots, i_k) \circ (j_1, \ldots, j_l)(x) = \begin{cases} j_{t+1} \text{ if } x = j_t \\ i_{r+1} \text{ if } x = i_r \\ x \text{ if } x \notin \{i_1, i_2, \ldots, i_k\} \cup \{j_1, \ldots, j_l\} \end{cases}$$

which is the same as $(j_1, \ldots, j_l) \circ (i_1, \ldots, i_k)(x)$.

Cycles can be used to represent all permutations in S_n.

Theorem

Every permutation in S_n is a product of disjoint cycles.

In lieu of a formal proof, let's consider an example to see how this works.

Say $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 5 & 7 & 1 & 4 & 2 & 8 \end{pmatrix}$ then to compute the cycles σ gives rise to we start with say 1 and see what happens when we keep applying σ:

- $\sigma(1) = 3$
- $\sigma(3) = 5$
- $\sigma(5) = 1$

so we stop and see that we get the cycle $(1, 3, 5)$.

Now consider another element of $\{1, 2, \ldots, 7, 8\}$, say 2, and see what repeated applications of σ do

- $\sigma(2) = 6$
- $\sigma(6) = 4$
- $\sigma(4) = 7$
- $\sigma(7) = 2$

and we stop and see that we get the cycle $(2, 6, 4, 7)$.

The only remaining element is 8, but $\sigma(8) = 8$ so we can ignore it.

So

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 5 & 7 & 1 & 4 & 2 & 8 \end{pmatrix}$$

can be written (compactly!) as the product

$$\sigma = (1,3,5)(2,6,4,7)$$

which conveys exactly the same information as the table notation.

Note, what if we had started the analysis by following what happens with 2, and then 1.
We would have gotten the cycles $(2,6,4,7)$ and $(1,3,5)$ and written $(2,6,4,7)(1,3,5)$ which, as we mentioned earlier, is the same permutation since disjoint cycles commute.

As another example, for practice, consider

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 7 & 5 & 3 & 8 & 2 & 6 & 4 & 9 \end{pmatrix}$$

which can be written as the product $(2,7,6)(3,5,8,4)$.

Note also, in a given cycle, the order doesn't matter as far as the cycle is concerned. e.g.

$$\begin{aligned}&(1,4,7,3,2) \in S_7\\ =&(4,7,3,2,1)\\ =&(7,3,2,1,4)\\ =&(3,2,1,4,7)\\ =&(2,1,4,7,3)\end{aligned}$$

More on Cycle Notation

If one has a given cycle

$$\sigma = (i_1, i_2, \ldots, i_k)$$

then the inverse image is easy to compute, namely

$$\sigma^{-1} = (i_1, i_k, i_{k-1}, \ldots, i_2)$$

for example $(1,4,7,3,2)^{-1} = (1,2,3,7,4)$.

Note: In a given cycle, we never include elements which are left fixed, by that cycle, e.g. $\sigma = (1,3,4) \in S_4$ fixes 2 and we purposely do **not** write this as $(1,3,4)(2)$ which would *technically* be correct, but is unnecessary, since the omission of 2 tells us that it is not 'moved' by σ.

Note also that the identity permutation is traditionally written as an 'empty' cycle, namely ().

Two other basic, but important facts come from using cycle notation:

Proposition

For $(i_1,\ldots,i_k)$ *we have* $|(i_1,\ldots,i_k)| = k$.

For example, for $\sigma = (1,3,5)$,

$$\sigma(1) = 3, \sigma(\sigma(1)) = \sigma(3) = 5, \text{ and } \sigma(\sigma(\sigma(1))) = \sigma(5) = 1$$

i.e. $\sigma^3(1) = 1$, and similarly $\sigma^3(3) = 3$ and $\sigma^3(5) = 5$, so $\sigma^3 = I$.

Proposition

If $\sigma = \sigma_1\sigma_2\cdots\sigma_t$ where the σ_i are disjoint cycles where σ_i is a k_i-cycle (of length/order k_i) then $|\sigma| = lcm(k_1, k_2, \ldots, k_t)$

Proof.

Suppose $g_1, g_2 \in G$ where $|g_1| = k_1$ and $|g_2| = k_2$ where $g_1g_2 = g_2g_1$ then $(g_1g_2)^2 = g_1g_2g_1g_2 = g_1g_1g_2g_2 = g_1^2g_2^2$.

And in general $(g_1g_2)^m = g_1^mg_2^m$, so the question is what is $|g_1g_2|$?

Well, $(g_1g_2)^m = g_1^mg_2^m$ so if $(g_1g_2)^m = e$ then $g_1^m = e$ and $g_2^m = e$ which means $|g_1|\big|m$ and $|g_2|\big|m$ so the least m for which this is true is $lcm(|g_1|, |g_2|) = lcm(k_1, k_2)$.

The argument is the same for the product of more than 2 group elements which commute, e.g. disjoint cycles in S_n. □

One kind of neat application of this is this, what is the largest order of any element in S_n?

Example: S_2 ? Well S_2 has just $2! = 2$ elements, namely $\{(), (1, 2)\}$, so the maximum order is 2.

Example: S_3 ? Since there are no '1-cycles' the only elements are 2-cycles (a, b) and 3-cycles (a, b, c) so the max. order is 3.

Example: S_4 ? In S_4 we have 4-cycles, 3-cycles, 2-cycles and products of two 2-cycles, $(a, b)(c, d)$ so the max is 4.

Example: S_5 ? In S_5 we have elements with cycle structures like (a, b, c, d, e), (a, b, c, d), (a, b, c), $(a, b)(c, d)$, (a, b)... any others?

Yes, S_5 has 'room' for a product $(a, b, c)(d, e)$ (where a, b, c, d, e are distinct!) and an element like this has order $lcm(|(a, b, c)|, |(d, e)|) = lcm(3, 2) = 3 \cdot 2 = 6$ since $gcd(3, 2) = 1$.

Example: S_6 ? The max order is 6. (Exercise.)

Example: S_7 ? The general principle to mention for this and all larger ones is the question of what kind of products of disjoint cycles can you make?

And therefore how big an *lcm* can you have?

This is also tied in with a question we shall examine later on, which is, how many ways can one write a natural number as a sum of natural numbers? Each such sum is called a 'partition' of n, for example:

$$\begin{aligned} 5 &= 5 \\ &= 4+1 \\ &= 3+2 \\ &= 3+1+1 \\ &= 2+2+1 \\ &= 2+1+1+1 \\ &= 1+1+1+1+1 \end{aligned}$$

and this ties in very closely with how many products of disjoint cycles exist in a given S_n.