MA294 Lecture

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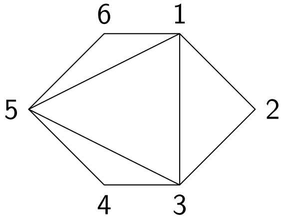
Graph Automorphisms

Definition

A graph Γ consists of two sets, $V = \{v_1, \ldots, v_n\}$, **vertices** (or **nodes**) and **edges** $E = \{[v_i, v_j]\}$ which join different pairs of nodes.

And for the sake of simplicity, we can simply refer to the vertices by their numbers, i.e. i instead of v_i

For example:



A graph is a means of depicting a set of relationships that exists between different elements of the set of vertices.

A natural (although huge!) example of this is the internet, where the nodes are computers and edges are the existence (or non-existence) of connections between the computers.

Our examples will be small obviously, and in particular we are interested in ways to permute the vertices of the graph but not change the essential 'connectivity' of the graph.

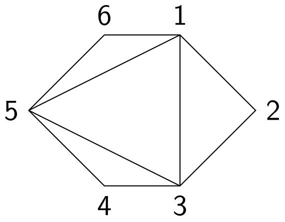
We'll make this definition precise in a moment.

To start with, we need a definition.

Definition

Given a graph $\Gamma = (V, E)$ the degree of any vertex v ($deg(v)$) is the number of edges connected to that vertex.

So for our first example:



we have $deg(1) = 4$, $deg(2) = 2$, $deg(3) = 4$, $deg(4) = 2$, $deg(5) = 4$, and $deg(6) = 2$.

Definition

A graph automorphism of a graph $\Gamma = (V, E)$ is an element $\sigma \in Perm(V)$ such that if i and j are connected by k edges, then $\sigma(i)$ and $\sigma(j)$ are too.

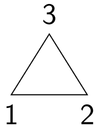
So for the example above, where $V = \{1, 2, 3, 4, 5, 6\}$ we can view the automorphism group as a subgroup of S_6, the question is, what are the elements of this group?

To begin with, we have the following very useful fact regarding degrees.

Proposition

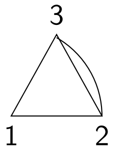
If $\Gamma = (V, E)$ *and* $\sigma \in Aut(\Gamma)$ *is a graph automorphism, then for* $v \in V$, $deg(\sigma(v)) = deg(v)$.

Before we look at our original example, let's look at some simpler ones.



Here $V = \{1, 2, 3\}$ and each node is connected to every other by one edge, so $Aut(\Gamma) = S_3$, since every permutation in S_3 preserves the connectivity of the edges.

What if we modify this example slightly.



And we see now that $deg(2) = 3$ and $deg(3) = 3$ while $deg(1) = 2$ which means that if $\sigma \in Aut(\Gamma)$ then $\sigma(1) = 1$, which severely limits the possible choices for σ, in fact to just two.

That is $Aut(\Gamma) = \{(), (2,3)\}$ where, indeed, vertices 2 and 3 are still connected by two edges, even after we swap them.

To help simplify the analysis a bit we consider a class of graphs whose structure is relatively straightforward.

Specically, many important graph examples have the property that a given pair of vertices are connected by **at most** 1 edge.

In this case, we can give a somewhat simpler version of the definition of a graph automorphism.

If $\sigma \in Aut(\Gamma)$ then $\sigma(i)$ and $\sigma(j)$ are connected by an edge if and only if i and j are connected.

As such, if each node is connected to another by at most one edge, we can enumerate the edge set E explicitly as 'ordered pairs' $[i,j]$ if i and j are connected.

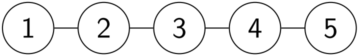
And so $[i,j] \in E$ if and only if $[\sigma(i),\sigma(j)] \in E$.

Note, we regard $[i,j]$ and $[j,i]$ as being the same.

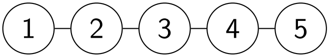
As such, $\sigma \in Aut(\Gamma)$ maps E to itself!

If we connect together n-vertices in a row, we get the 'Path Graph' P_n.

For example $n = 5$ gives P_5 (where we are emphasing the vertices)

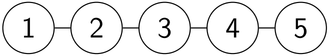


What is the automorphism group of P_5? (or any P_n?)



Clearly $Aut(P_5) \leq S_5$ but if $\sigma \in Aut(P_5)$, what is $\sigma(1)$ for example?

To keep track of what's happening, let's enumerate the edges.

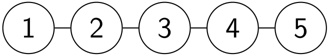


Here $E = \{[1,2],[2,3],[3,4],[4,5]\}$

Recall then that $[\sigma(i),\sigma(j)] \in E$ if and only $[i,j] \in E$.

So suppose $\sigma(1) = 2$ for example, then the edge $[1,2]$ is sent to the edge $[\sigma(1),\sigma(2)] = [2,\sigma(2)]$.

But the only edges in E which contain '2' are $[1,2]$ and $[2,3]$.

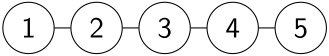


$E = \{[1,2],[2,3],[3,4],[4,5]\}$

So $[1,2] \mapsto [2,\sigma(2)]$ where $[2,\sigma(2)] = [2,1] = [1,2]$ or $[2,3]$.
This implies that either $\sigma(2) = 1$ or $\sigma(2) = 3$.

If $\sigma(2) = 1$ then $[1,2] \mapsto [1,2]$ and $[2,3] \mapsto [1,\sigma(3)]$ which creates a problem.

The only edge containing 1 is $[1,2]$ which would imply that $\sigma(3) = 2$, but $\sigma(1) = 2$.



$E = \{[1,2],[2,3],[3,4],[4,5]\}$

As such $\sigma(1) = 2$ implies that $\sigma(2) = 3$, so that $[1,2] \mapsto [2,3]$ which in turn implies that $[2,3] \mapsto [3,\sigma(3)]$ but the only edges containing 3 are $[2,3]$ and $[3,4]$.

As such, either $\sigma(3) = 2$ or $\sigma(3) = 4$, but $\sigma(3) = 2$ is impossible because $\sigma(1) = 2$.

So.. $\sigma(3) = 4$, and thus we have (so far) that $\sigma(1) = 2$, $\sigma(2) = 3$ and $\sigma(3) = 4$ which means that $[1,2] \mapsto [2,3]$ and $[2,3] \mapsto [3,4]$.

This leaves $\sigma(4)$ and $\sigma(5)$.

But now we can appeal to cycle structure, since we have that $\sigma(1) = 2$, $\sigma(2) = 3$, and $\sigma(3) = 4$ so as a product of cycles, we have that

$$\sigma = (1, 2, 3, 4...$$

and since σ must be a product of disjoint cycles, either $\sigma(4) = 1$ whence $\sigma(5) = 5$ which means

$$\sigma = (1, 2, 3, 4)$$

or $\sigma(4) = 5$ and $\sigma(5) = 1$ so that

$$\sigma = (1, 2, 3, 4, 5)$$

.However, neither of these is permissible based on the edges.

$$E = \{[1,2],[2,3],[3,4],[4,5]\}$$
$$\sigma = (1,2,3,4)$$

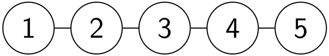
implies that $[1,2] \mapsto [2,3]$, $[2,3] \mapsto [3,4]$ and $[3,4] \mapsto [4,1]$... but $[4,1] = [1,4] \notin E$!

And

$$E = \{[1,2],[2,3],[3,4],[4,5]\}$$
$$\sigma = (1,2,3,4,5)$$

implies that $[1,2] \mapsto [2,3]$, $[2,3] \mapsto [3,4]$ and $[3,4] \mapsto [4,5]$ and $[4,5] \mapsto [5,1]$ but $[5,1] = [1,5] \notin E$!

So.. $\sigma(1) = 2$ is just impossible.

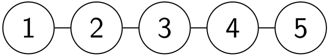


$E = \{[1,2],[2,3],[3,4],[4,5]\}$

So what now for $\sigma(1)$?

Well, if we look at the degrees of the vertices we notice that $deg(1) = 1$ and $deg(5) = 1$ while the other nodes have degree 2. As such $\sigma(1) = 1$ or $\sigma(1) = 5$.

If $\sigma(1) = 1$ then since vertex 1 and 2 are connected by an edge, then $\sigma(2)$ must be a vertex connected to $\sigma(1) = 1$ which means $\sigma(2) = 2$, and similarly $\sigma(3) = 3$ and $\sigma(4) = 4$ and $\sigma(5) = 5$ so that $\sigma = I$.



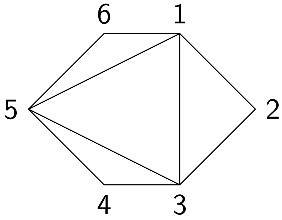
If $\sigma(1) = 5$ then since 1 and 2 are connected by an edge, then $\sigma(2) = 4$ by necessity, and symmetrically $\sigma(4) = 2$ which means $\sigma(3) = 3$.

As such $Aut(P_5) = \{I, (1,5)(2,4)\} \cong C_2$.

Question: What about other P_n?

Note also that 3 in P_5 was fixed, basically because it was in the middle, what if n is even?

So for the earlier graph Γ



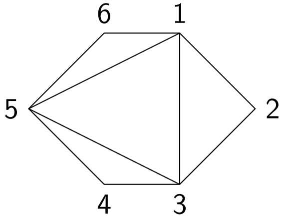
we have that

$$deg(1) = deg(3) = deg(5) = 4$$

and

$$deg(2) = deg(4) = deg(6) = 2$$

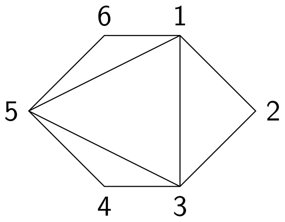
so if $i \in \{1,3,5\}$ then $\sigma(i) \in \{1,3,5\}$ and similarly if $i \in \{2,4,6\}$ then $\sigma(i) \in \{2,4,6\}$ too.



We will use the notation $a \leftrightarrow b$ to indicate vertices a and b are connected.

If $\sigma(1) = 3$ and $\sigma(3) = 1$ then since $1 \leftrightarrow 2$ and $3 \leftrightarrow 2$, we must have $\sigma(2) = 2$.

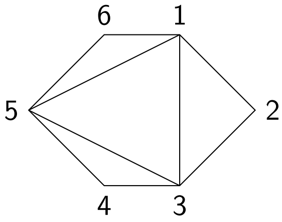
And since $deg(5) = 4$ then $\sigma(5) = 5$, and since $1 \leftrightarrow 6$ and $3 \leftrightarrow 4$ we must have $\sigma(4) = 6$ and $\sigma(6) = 4$, ergo $(1,3)(4,6) \in Aut(\Gamma)$.



Similarly, if $\sigma(1) = 5$ and $\sigma(5) = 1$ then $\sigma(3) = 3$ since $deg(3) = 4$.

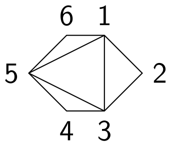
But since $1 \leftrightarrow 6$ and $5 \leftrightarrow 6$ we have $\sigma(6) = 6$, and since $1 \leftrightarrow 2$ and $5 \leftrightarrow 4$ we must have $\sigma(2) = 4$ and $\sigma(4) = 2$.

As such $(1,5)(2,4) \in Aut(\Gamma)$, and a similar argument involving the nodes 3 and 5 implies that $(3,5)(2,6) \in Aut(\Gamma)$.



Now suppose $\sigma(1) = 3$, $\sigma(3) = 5$ and $\sigma(5) = 1$ then since $1 \leftrightarrow 6$ and $1 \leftrightarrow 2$, and $3 \leftrightarrow 2$ and $3 \leftrightarrow 4$, and $5 \leftrightarrow 6$ and $5 \leftrightarrow 4$ we must also cyclically rotate the vertices $2, 4, 6$ in the same clockwise way as the vertices $1, 3, 5$.

As such $(1, 3, 5)(2, 4, 6) \in Aut(\Gamma)$, and similarly $(1, 5, 3)(2, 6, 4) \in Aut(\Gamma)$.



So we've established that $Aut(\Gamma)$ contains

$$\{(), (1,3)(4,6), (1,5)(2,4), (3,5)(2,6), (1,3,5)(2,4,6), (1,5,3)(2,6,4)\}$$

but does it contain any other elements?

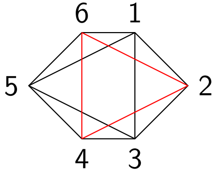
No, because we were bound by the possibilities for where the nodes 1, 3, and 5 could be sent, and each such possibility forces the movements of $2, 4, 6$ so there are no more.

The last detail to consider is whether $Aut(\Gamma)$, equal to

$$\{(),(1,3)(4,6),(1,5)(2,4),(3,5)(2,6),(1,3,5)(2,4,6),(1,5,3)(2,6,4)\}$$

is isomorphic to something 'familiar'?

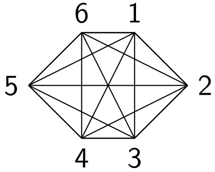
Yes, $Aut(\Gamma) \cong D_3$, and the idea is to look at the two 'triangles'



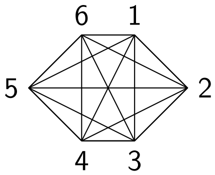
one of which is part of the graph $\{1, 3, 5\}$ and the other there by virtue of being attached to these, namely $\{2, 4, 6\}$,
i.e.$\{(), (1, 3), (1, 5), (3, 5), (1, 3, 5), (1, 5, 3)\} \cong D_3$ and
$\{(), (4, 6), (2, 4), (2, 6), (2, 4, 6), (2, 6, 4)\} \cong D_3$.

Complete Graphs

Here is a kind of 'extreme' example, although one which is actually pretty simple in a certain sense.



One sees that for each $v \in V = \{1, 2, 3, 4, 5, 6\}$ that $deg(v) = 5$ and, more to the point, each vertex is connected to every other vertex by an edge.



So for any $\sigma \in S_6$, one has that $\sigma(i)$ and $\sigma(j)$ are connected since 'i' and 'j' are connected.

As such $Aut(\Gamma)$ is all of S_6 and we have an example of a complete graph namely one for which each vertex is connected to every other vertex.

And for each n we have such a graph, and each has automorphism group equal to the entirety of S_n.