# MA294 Lecture

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# Orbits and Stabilizers

Symmetric groups also help us solve certain combinatorial (counting) problems.

Let $X$ be a finite set and $G \leq Perm(X)$ and define an equivalence relation $\sim$ on $X$ as follows:

$$x \sim y \text{ if } g(x) = y \text{ for some } g \in G$$

Let's verify this is an equivalence.

- $x \sim x$ since $e(x) = x$ for $e \in G$ the identity element which fixes every element of $X$
- If $x \sim y$ then $g(x) = y$ for some $g \in G$, but then $x = g^{-1}(g(x)) = g^{-1}(y)$, i.e. $g^{-1}(y) = x$ so $y \sim x$
- If $x \sim y$ and $y \sim z$ then $g(y) = x$ and $g'(z) = y$ which means $g(g'(z)) = x$, i.e. $(g \circ g')(z) = x$ so $z \sim x$

## Definition

For this equivalence relation, the equivalence classes are called the orbits with respect to the action of $G$ on $X$.

Specifically, for $x \in X$, let $Gx = \{g(x) \mid g \in G\}$ where, if $G = \{g_1, \dots, g_n\}$ is literally $\{g_1(x), g_2(x), \dots, g_n(x)\}$

Note, it *is* possible that $g_i(x) = g_j(x)$ for two *distinct* elements $g_i, g_j \in G$, including the possibility that $g(x) = x$ for some $g \in G$ where $g \neq e$.

Example: $G = \{(), (1,2)(3,4,5), (3,5,4), (1,2), (1,2)(3,5,4), (3,4,5)\}$

- $G1 = \{1, 2\}$
- $G2 = \{1, 2\}$
- $G3 = \{3, 4, 5\}$
- $G4 = \{3, 4, 5\}$
- $G5 = \{3, 4, 5\}$

and we see here another phenomenon, namely that $Gx = Gy$ even when $x \neq y$.

Of particular importance for applications is the determination of $|Gx|$ for each $x \in X$.

**Definition**

For $G \leq Perm(X)$, and $x, y \in X$ let

$$G(x \to y) = \{g \in G \mid g(x) = y\}$$

and for $x = y$ we have

$$G(x \to x) = \{g \in G \mid g(x) = x\}$$

where $G(x \to x)$ is the stabilizer of $x$ in $G$, which we also denote $G_x$.

Note, for any $x \in X$, we have $G_x \leq G$.

Why? If $g_1, g_2 \in G_x$ then $(g_1 \circ g_2)(x) = g_1(g_2(x)) = g_1(x) = x$, and if $g(x) = x$ then $g^{-1}(g(x)) = g^{-1}(x)$, that is $x = g^{-1}(x)$ and thus $g^{-1} \in G_x$ too.

We observed that $G_x = G(x \to x)$ is a subgroup of $G$ for each $x \in X$, so what about $G(x \to y)$ for $x \neq y$?

## Theorem

*Let $G \leq Perm(X)$ and let $h \in G(x \to y)$ then*

$$G(x \to y) = hG_x$$

*a left coset of $G_x$ in $G$.*

## Proof.

If $g \in G_x$ and $h(x) = y$ then $(h \circ g)(x) = h(g(x)) = h(x) = y$ so that $h \circ g \in G(x \to y)$ so $hG_x \subseteq G(x \to y)$.

If $h \in G(x \to y)$ then $h = (h \circ e) \in hG_x$ since certainly $e \in G_x$ and so $G(x \to y) \subseteq hG_x$, so $G(x \to y) = hG_x$. □

As a consequence we have the following really fundamental fact about permutation groups, called the Orbit-Stabilizer Theorem.

## Theorem

*If* $G \leq Perm(X)$ *then* $|Gx| = [G : G_x] = \frac{|G|}{|G_x|}$.

## Proof.

The basic idea is that if $Gx = \{y_1, \ldots, y_m\}$ then $G(x \to y_i) \cap G(x \to y_j) = \emptyset$ for $i \neq j$ and $G(x \to y_i) = h_i G_x$ for distinct $\{h_1, \ldots, h_m\}$ where $G = h_1 G_x \cup h_2 G_x \cup \cdots \cup h_m G_x$.

But this says that '$m$' which is $|Gx|$ is the same as the number of distinct cosets of $G_x$ in $G$, that is $[G : G_x]$ where $[G : G_x] = \frac{|G|}{|G_x|}$. □

Example: $G = \langle (1,2)(3,4,5) \rangle =$
$\{(), (1,2)(3,4,5), (3,5,4), (1,2), (1,2)(3,5,4), (3,4,5)\}$

- $|G1| = 2 \leftrightarrow G_1 = \langle (3,4,5) \rangle \leftrightarrow [G : G_1] = \frac{6}{3} = 2$
- $|G3| = 3 \leftrightarrow G_3 = \langle (1,2) \rangle \leftrightarrow [G : G_3] = \frac{6}{2} = 3$

The main point of this theorem is that one may determine $|Gx|$ (which may not be easy to find) by using $G_x$ which is generally easier to determine.

For a given $G \leq Perm(X)$ a related question that is also important is to find out how many distinct orbits there are.

For example, with $G = \langle(1,2)(3,4,5)\rangle$, we saw that $G1 = \{1,2\}$, $G2 = \{1,2\}$ while $G3 = \{3,4,5\}$, $G4 = \{3,4,5\}$, and $G5 = \{3,4,5\}$ as well.

**Definition**

For $G \leq Perm(X)$ and $g \in G$, let $F(g) = \{x \in X \mid g(x) = x\}$ which are the set of those elements of $G$ which fix $x$.

## Theorem (Frobenius)

*For $G \leq Perm(X)$ the number of orbits of $G$ on $X$ is*

$$\frac{1}{|G|}\sum_{g\in G}|F(g)|$$

*which is a whole number.*

Before exploring the proof, we should mention that this result is actually the tool that we will use for solving different combinatorial (counting) problems.

It is built on the Orbit-Stabilizer Theorem, but is different in that it actually tells us how many distinct orbits there are, which is actually a bit more mysterious than the size of the orbit of a particular element of $x$.

This theorem is telling us a *lot* about the action of $G$ on $X$ overall.

PROOF:
Consider the following subset of $G \times X$

$$E = \{(g, x) \mid g(x) = x\}$$

whose size we will compute in two different ways, and these two ways of counting will give us the statement of the theorem.

If $G = \{g_1, \ldots, g_n\}$ and $X = \{x_1, \ldots, x_r\}$ then $E = E_{g_1} \cup \cdots \cup E_{g_n}$ where

$$E_{g_i} = \{(g_i, x) \in G \times X \mid (g_i(x) = x\}$$

where the $E_{g_i}$ are disjoint, and, for some $g_i$, potentially empty.
Moreover, it's clear that $E_{g_i} = F(g_i)$.

So we have that $|E| = \sum_{i=1}^{n} |F(g_i)|$.

Another way to view $E$ is as follows:

$$E = E_{x_1} \cup E_{x_2} \cup \cdots \cup E_{x_r}$$

where $E_{x_j} = \{(g, x_j) \in G \times X \mid g(x_j) = x_j\}$ and so $|E_{x_j}| = |G_{x_j}|$ (!), so we have that

$$|E| = \sum_{j=1}^{r} |G_{x_j}|$$

So if $\{x_{11},\ldots,x_{1e_1}\}$, $\{x_{21},\ldots,x_{2e_2}\}$,...,$\{x_{t1},\ldots,x_{te_t}\}$ are the distinct orbits of $G$ on $X$, namely $t$ orbits, where the size of the $i^{th}$ orbit is $e_i$.

As such $|Gx_{ij}| = e_i$ for $i = 1,\ldots,t$ and $j = 1,\ldots,e_t$, so $\frac{|G|}{|G_{x_{ij}}|} = e_i$ and so $|G_{x_{ij}}| = \frac{|G|}{e_i}$ for $i = 1,\ldots,t$.

Thus

$$\begin{aligned}\sum_{x\in X}|G_x| &= e_1\frac{|G|}{e_1} + e_2\frac{|G|}{e_2} + \cdots + e_t\frac{|G|}{e_t}\\ &= t|G|\end{aligned}$$

and so... $\sum_{g\in G}|F(g)| = \sum_{x\in X}|G_x| = t|G|$ which means

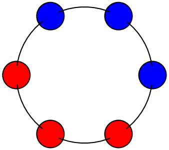
$$t = \frac{1}{|G|}\sum_{g\in G}|F(g)|$$

as claimed. □

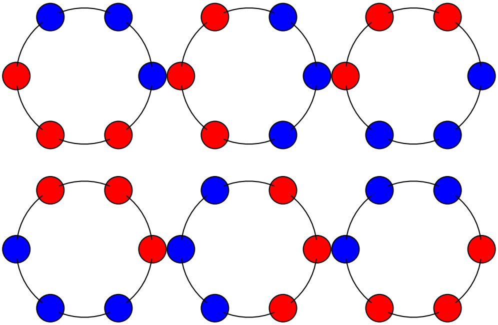
# Applications to Counting

Say 6 beads are strung together in circle, where 3 are red, and 3 are blue.

For example:

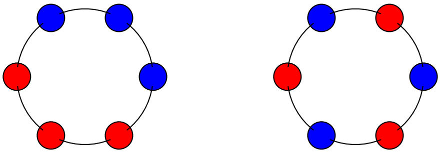


We shall consider all of these equivalent configurations:



And all of these can be considered in the same orbit with resepct to the action of $D_6$ since the positions of the beads can be viewed as at the corners of a hexagon. (We see the effect of 'rotating' in 60 degree increments, but even if we include the 'flips' of the hexagon, the result is an orbit with six configurations above.)

However, these two configurations are not in the same orbit:



This can be seen by realizing that no flip or rotation will change the fact that the colors in one configuration alternate, whereas in the other they are clustered together.

What we want is to count how many distinct orbits there are with respect to the action of $D_6$.

We will utilize the previous theorem, namely we will determine $|F(g)|$ for each $g \in D_6$.

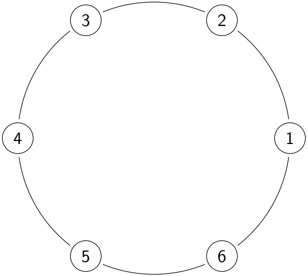
First, we need to consider how many total configurations of 3 blue and 3 red beads there are, before we can subdivide them into orbits.

Since there are 6 beads total, and by choosing 3 red ones, we also determine which are blue then we have

$$\binom{6}{3} = \frac{6!}{3! \cdot 3!} = 20$$

configurations total.

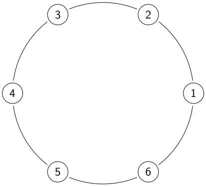
For the 'hexagon' of vertices, we define
$D_6 = \{r_0, r_{60}, r_{120}, r_{180}, r_{240}, r_{300}, f_{(1,2)}, f_{(2,3)}, f_{(3,4)}, f_{(1,4)}, f_{(2,5)}, f_{(3,6)}\}$



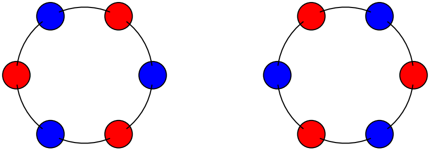

So we shall compute $|F(g)|$ for each $g \in D_6$.

$|F(r_0)| = 20$ Why? Well the identity obviously fixes every configuration.

$|F(r_{60})| = 0$ Why? Well no matter how the six beads are distributed, a $60^\circ$ rotation will change at least one position from red to blue.

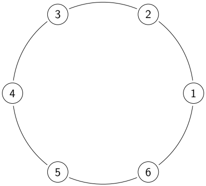



$|F(r_{120})| = 2$ Why? Well, a $120^\circ$ rotation pushes every bead forward two positions, so the question is whether there is a way to distribute the beads so that a two position turn leaves the colors unchanged. Yes.

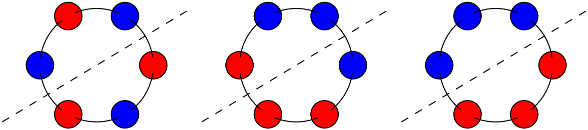


In a similar way we find that $|F(r_{180})| = 0$, $|F(r_{240})| = 2$, $|F(r_{300})| = 0$.

So what about flips?



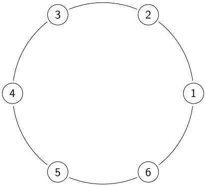

$|F(f_{(1,2)})| = 0$ Why? Consider these arrangements.



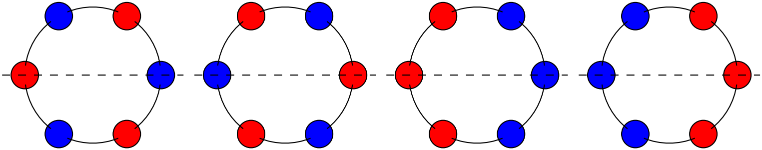
i.e. The distribution of colors on either side of the line will never be the same after the flip.

In a similar way, we find that flips about a line passing through opposite sides of the hexagon leave no configurations unchanged, i.e.

$|F(f_{(2,3)})| = 0$, and $|F(f_{(3,4)})| = 0$,




For the flips through lines connecting opposite vertices, $f_{(1,4)}$, $f_{(2,5)}$ and $f_{(3,6)}$ the behavior is different.
$|F(f_{(1,4)})| = 4$, here they are.



And similarly $|F(f_{(2,5)})| = 4$ and $|F(f_{(3,6)})| = 4$.

So, in the final analysis, the number of orbits is

$$\frac{1}{12}\left(20+2+2+4+4+4\right)=\frac{1}{12}(36)=3$$

which are the orbits of these 'distinct' configurations: