MA294 Lecture

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April 16, 2024

Generating Functions

Sequences $\{u_n\}$ for $n = 0, 1, \ldots$, particularly recursively defined ones, can be understood and analyzed by using the terms in the sequence as coefficients of an infinite series

$$U(x) = u_0 + u_1 x + u_2 x^2 + \ldots$$

which is called the (ordinary) generating function for $\{u_n\}$.

We should note that $U(x)$ is not really a function, and again, we are avoiding discussions of convergence as they are irrelevent.

The principle idea is the following:

(i) Take the series $U(x)$ and utilize the 'recurrence relation' (namely the rule which determines how a given u_n depends on the previous terms u_i for $i < n$), to find an equation that $U(x)$ satisfies.

(ii) Use this equation to come up with an 'explicit', that is *non-recursive* formula for the terms $\{u_n\}$. That is, we want to find an explicit function $f(n)$ for $n = 0, 1, \ldots$, such that $u_n = f(n)$.

(iii) The basis for this function is that the equation $U(x)$ satisfies, gives rise to a power series, similar to the geometric (or other) series for example, whose determination will require a fair amount of algebra, including some of the facts about the partial fractions decomposition of a rational function.

The prime example of such a sequence that can be analyze this way is the **Fibonacci Sequence** $\{f_n\}$ defined as follows:

$$f_0 = 0 \ f_1 = 1, f_{n+1} = f_n + f_{n-1} \text{ for } n > 1$$

so in particular $f_2 = f_1 + f_0 = 1 + 0 = 1$, $f_3 = f_2 + f_1 = 1 + 1 = 2$, namely $\{0, 1, 1, 2, 3, 5, 8, 13, \dots\}$.

For sequences like the Fibonacci sequence, our goal is to be able to determine the n-the Fibonacci number f_n *without* needing to compute all the f_i for $i < n$. (i.e. what is termed a *closed formula* for f_n)

Without going into too much detail, suffice it to say that the Fibonacci sequence is ubiquitous throughout many areas of mathematics.

As it is a fairly detailed calculation, we shall return to the Fibonacci numbers after examining a number of more modest examples first, to get a feel for the techniques involved.

Consider the sequence $\{u_n\}$ defined by $u_0 = 1$, $u_n = 3u_{n-1}$ where now

$$U(x) = u_0 + u_1x + u_2x^2 + u_3x^3 + u_4x^4 + \dots$$

so the question is, what 'equation' does $U(x)$ satisfy? Observe:

$$\begin{aligned} U(x) &= u_0 + u_1x + u_2x^2 + u_3x^3 + u_4x^4 + \dots \\ &= u_0 + (3u_0)x + (3u_1)x^2 + (3u_2)x^3 + (3u_3)x^4 + \dots \\ &= u_0 + (3u_0x + 3u_1x^2 + 3u_2x^3 + 3u_3x^4 + \dots) \\ &= u_0 + 3(u_0x + u_1x^2 + u_2x^3 + u_3x^4 + \dots) \\ &= u_0 + 3x(u_0 + u_1x + u_2x^2 + u_3x^3 + \dots) \\ &= 1 + 3xU(x) \end{aligned}$$

which can be solved to yield $U(x) = \frac{1}{1-3x}$

So now we know that $U(x) = u_0 + u_1 x + \cdots = \frac{1}{1-3x}$ but the key fact which we invoke now is that $\frac{1}{1-3x}$ is expressible as a series, indeed a geometric series.

That is, we know

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

So by replacing 'x' by $3x$ above we get:

$$\begin{aligned}\frac{1}{1-3x} &= 1 + 3x + (3x)^2 + (3x)^3 + \ldots \\ &= 1 + 3x + 3^2x^2 + 3^3x^3 + \ldots \\ &= \sum_{n=0}^{\infty} 3^n x^n\end{aligned}$$

So we now bridge these two expressions:

$$U(x) = \sum_{n=0}^{\infty} u_n x^n$$
$$= \sum_{n=0}^{\infty} 3^n x^n$$

the conclusion being that $u_n = 3^n$ for *all* n.

This is not a super difficult example in that one *could* observe the pattern that $u_0 = 1$, $u_1 = 3 \cdot 1 = 3$, so $u_2 = 3u_1 = 3 \cdot 3 = 3^2$, etc. but as we'll see, there are more complicated examples where simply computing 'the first few terms' is not quite sufficient to get the explicit formula for u_n for *all* n.

More Generating Function Examples

A simple way to increase the complexity is to add an extra 'constant' term to the recurrence.

Let's take our last example and let $u_0 = 1$ as before, but define $u_n = 3u_{n-1} + 1$ which will be a *different* sequence, but not simply $u_n = 3^n + 1$ (which **isn't** the formula).

Again, we write down a $U(x)$ series and try to determine an equation it satisfies.

$$
\begin{aligned}
U(x) &= u_0 + u_1 x + u_2 x^2 + u_3 x^3 + u_4 x^4 + \dots \\
&= u_0 + (3u_0 + 1)x + (3u_1 + 1)x^2 + (3u_2 + 1)x^3 + (3u_3 + 1)x^4 + \dots \\
&= u_0 + (3u_0 x + 3u_1 x^2 + 3u_2 x^3 + 3u_3 x^4 + \dots) + (x + x^2 + x^3 + x^4 + \dots \\
&= u_0 + 3x(u_0 + u_1 x + u_2 x^2 + u_3 x^3 + \dots) + (x + x^2 + x^3 + x^4 + \dots) \\
&= 1 + 3xU(x) + \left[1 + x + x^2 + x^3 + \dots\right] - 1
\end{aligned}
$$

which simplifies to $U(x) = 1 + 3xU(x) + \frac{1}{1-x} - 1$.

Solving for $U(x)$ yields

$$U(x)(1-3x) = 1 + \frac{1}{1-x} - 1$$

$$\downarrow$$

$$U(x) = \frac{1}{(1-3x)(1-x)}$$

So let's determine the coefficients in the series for $U(x)$, based on the series representations of $\frac{1}{1-3x}$ and $\frac{1}{1-x}$.

Using the partial fractions theorem there are constants A, B such that:

$$U(x) = \frac{1}{(1-3x)(1-x)} = \frac{A}{1-3x} + \frac{B}{1-x}$$

which can be solved to yield $A = \frac{3}{2}$, $B = -\frac{1}{2}$ which implies

$$U(x) = \frac{3}{2}\sum_{n=0}^{\infty}(3x)^n - \frac{1}{2}\sum_{n=0}^{\infty}x^n$$

which, since $U(x) = u_0 + u_1x + \ldots$ implies that $u_n = \frac{3}{2}3^n - \frac{1}{2}$ for each n.

So we've determined that $u_n = \frac{3}{2}3^n - \frac{1}{2}$ from this analysis.

It's still good to 'verify' that this is indeed the formula for the n^{th} term.

$u_0 = 1$ implies $u_1 = 3(u_0) + 1 = 4$ and by the forumala

$$u_1 = \frac{3}{2}3^1 - \frac{1}{2} = \frac{9}{2} - \frac{1}{2} = 4$$

etc.

Now let's get back to the Fibonacci sequence, which is a bit more involved, partly because the recrurrence $f_n = f_{n-1} + f_{n-2}$ involves three terms of the sequence as opposed to the two in the example, $u_n = 3u_{n-1}$.

$$
\begin{aligned}
F(x) &= f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \dots \\
&= f_0 + f_1 x + (f_0 + f_1)x^2 + (f_1 + f_2)x^3 + (f_2 + f_3)x^4 + \dots \\
&= f_0 + f_1 x + (f_0 x^2 + f_1 x^3 + f_2 x^4 + \dots) + (f_1 x^2 + f_2 x^3 + f_3 x^4 + \dots) \\
&= f_0 + f_1 x + x^2(f_0 + f_1 x + f_2 x^2 + \dots) + x(f_1 x + f_2 x^2 + f_3 x^3 + \dots) \\
&= f_0 + f_1 x + x^2 F(x) + x(F(x) - f_0)
\end{aligned}
$$

And if we recall that $f_0 = 0,\ f_1 = 1$ then this boils down to the formula

$$
\begin{gathered}
F(x) = x + x^2 F(x) + xF(x) \\
\downarrow \\
F(x) = \frac{x}{1 - x - x^2}
\end{gathered}
$$

As $F(x) = \frac{x}{1-x-x^2}$ we would like to utilize partial fractions to express this in terms of simple fractions which will yield geometric series of some sort.

The subtle part is factoring the denominator $1 - x - x^2$ as $(1 - ax)(1 - bx)$ and ultimately write

$$F(x) = \frac{A}{1 - ax} + \frac{B}{1 - bx}$$

for some constants A and B.

If $1 - x - x^2 = (1 - ax)(1 - bx) = 1 - (a + b)x + abx^2$ then $a + b = 1$ and $ab = -1$, facts we will use soon.

So
$(1-ax)(1-bx) = a(\frac{1}{a}-x)b(\frac{1}{b}-x) = ab(\frac{1}{a}-x)(\frac{1}{b}-x) = ab(x-\frac{1}{a})(x-\frac{1}{b})$
and since $ab = -1$ we get

$$1 - x - x^2 = (1 - ax)(1 - bx) = -(x - \frac{1}{a})(x - \frac{1}{b})$$

Now $1-x-x^2 = -x^2-x+1$ has roots given by the quadratic formula, namely

$$\frac{1\pm\sqrt{5}}{-2} = \frac{-1\pm\sqrt{5}}{2}$$

.

So since

$$1-x-x^2 = -(x-\frac{1}{a})(x-\frac{1}{b})$$

whose roots are $\frac{1}{a}$ and $\frac{1}{b}$ then we can assume that

$$\frac{1}{a} = \frac{-1+\sqrt{5}}{2} \text{ and } \frac{1}{b} = \frac{-1-\sqrt{5}}{2}$$

which means that

$$a = \frac{2}{-1+\sqrt{5}}$$

$$= \frac{1+\sqrt{5}}{2}$$

$$b = \frac{2}{-1-\sqrt{5}}$$

$$= \frac{1-\sqrt{5}}{2}$$

So..

$$\begin{aligned}\frac{x}{1-x-x^2} &= \frac{x}{(1-ax)(1-bx)} \\ &= \frac{A}{1-ax} + \frac{B}{1-bx} \\ &\downarrow \\ x &= A(1-bx) + B(1-ax)\end{aligned}$$

which can be solved to yield $A = \frac{1}{\sqrt{5}}$, $B = -\frac{1}{\sqrt{5}}$ so that

$$F(x) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} a^n x^n - \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} b^n x^n$$

which implies, ultimately that $f_n = \frac{a^n - b^n}{\sqrt{5}}$

Now, instead of 'a' and 'b' we write $\varphi = \frac{1+\sqrt{5}}{2}$ and $\bar{\varphi} = \frac{1-\sqrt{5}}{2}$ where we refer to φ is the 'golden mean' or 'golden ratio' which we'll talk about more in a moment.

So we've established that $f_n = \frac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}}$ for each $n \geq 0$ and given that $\varphi = \frac{1+\sqrt{5}}{2}$ and $\bar{\varphi} = \frac{1-\sqrt{5}}{2}$ it is perhaps not altogether obvious why

$$\frac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}}$$

is an integer, even though it *must* be as it does equal f_n and each f_n is patently a whole number.

The Golden Mean

Last time, we showed that for the Fibonacci Sequence $\{f_n\}$ defined by $f_0 = 1,\ f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$ that

$$f_n = \frac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}}$$

which is surprising given that $\varphi = \frac{1+\sqrt{5}}{2}$ and $\bar{\varphi} = \frac{1-\sqrt{5}}{2}$ are not whole numbers.

Nonetheless, this formula does reproduce the terms of f_n, for example:

$$\begin{aligned}
\frac{\varphi^0 - \bar{\varphi}^0}{\sqrt{5}} &= \frac{1-1}{\sqrt{5}} \\
&= 0 \\
\frac{\varphi^1 - \bar{\varphi}^1}{\sqrt{5}} &= \frac{\sqrt{5}}{\sqrt{5}} \\
&= 1 \\
\frac{\varphi^2 - \bar{\varphi}^2}{\sqrt{5}} &= \frac{\sqrt{5}}{\sqrt{5}} \\
&= 1 \\
\frac{\varphi^3 - \bar{\varphi}^3}{\sqrt{5}} &= \frac{2\sqrt{5}}{\sqrt{5}} \\
&= 2
\end{aligned}$$