MA294 Lecture

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Differentiation of Series

Another useful tool for working with infinite series, especially those arising from fractions $\frac{A(x)}{B(x)}$ is that we can 'formally' differentiate these series using repeated applications of the basic power rule

$$\frac{d}{dx}x^n = nx^{n-1}$$

which is well defined algebraically, even though we are not invoking any limits in the process, but rather using the formal properties of the differentiation operation.

Theorem

If $f(x) \in F[[x]]$ *where* $f(x)$ *is a polynomial which is invertible, then if*

$$\frac{1}{f(x)} = \sum_{n=0}^{\infty} c_n x^n$$

then

$$\begin{aligned}
\frac{d}{dx}\frac{1}{f(x)} &= \frac{d}{dx}\sum_{n=0}^{\infty} c_n x^n \\
&= \frac{d}{dx}(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots) \\
&= c_1 + 2c_2 x + 3c_2 x^2 + \dots \\
&= \sum_{n=1}^{\infty} n c_n x^{n-1}
\end{aligned}$$

That is, we can differentiate series in $F[[x]]$ and the 'function' represented by the derivative is given by the series obtained from differentiating the terms one-by-one.

This should be somewhat familiar if you recall the development of series in Calculus 2.

However, here we are still not assuming anything 'analytic' in the usage of this theorem, since it is actually a statement about algebra since the functions being represented by series are 'algebraic', i.e. polynomials or rational functions.

Example:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$$

$$\downarrow$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + \dots$$

$$\downarrow$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$$

To see that $1+2x+3x^2+\dots$ really *does* represent $\frac{1}{(1-x)^2}$ simply compute $(1-x)^2(1+2x+3x^2+\dots)$

$$
\begin{aligned}
(1-x)^2\left(\sum_{n=0}^{\infty}(n+1)x^n\right) &= (1-2x+x^2)\left(\sum_{n=0}^{\infty}(n+1)x^n\right)\\
&= \sum_{n=0}^{\infty}(n+1)x^n - 2x\left(\sum_{n=0}^{\infty}(n+1)x^n\right) + x^2\left(\sum_{n=0}^{\infty}(n+1)x^n\right)\\
&= \sum_{n=0}^{\infty}(n+1)x^n - \left(\sum_{n=0}^{\infty}(2n+2)x^{n+1}\right) + \left(\sum_{n=0}^{\infty}(n+1)x^{n+2}\right)\\
&= (1+2x+3x^2+4x^3+\dots)\\
&- (2x+4x^2+6x^3+\dots)\\
&+ (1x^2+2x^3+3x^4+\dots)\\
&= 1
\end{aligned}
$$

Here is a nice example with a sequence $\{u_n\}$ defined by the recurrence: $u_0 = 0,\ u_{n+1} = u_n + (n+1)$.

$$
\begin{aligned}
U(x) &= u_0 + u_1 x + u_2 x^2 + u_3 x^3 + \dots \\
&= u_0 + (u_0 + 1)x + (u_1 + 2)x^2 + (u_2 + 3)x^3 + \dots \\
&= u_0 + x(u_0 + u_1 x + \dots) + (x + 2x^2 + 3x^3 + \dots) \\
&= u_0 + x(u_0 + u_1 x + \dots) + x(1 + 2x + 3x^2 + \dots) \\
&= u_0 + xU(x) + x\left(\frac{1}{(1-x)^2}\right) \\
&= xU(x) + x\left(\frac{1}{(1-x)^2}\right)
\end{aligned}
$$

So we have that $U(x) - xU(x) = \dfrac{x}{(1-x)^2}$ which means $U(x) = \dfrac{x}{(1-x)^3}$.

The question is now, what is the series which represents $U(x)$?

Bear in mind that

$$\frac{d}{dx}\frac{1}{(1-x)^2} = \frac{2}{(1-x)^3}$$

so

$$\begin{aligned}
\frac{2}{(1-x)^3} &= \frac{d}{dx}\sum_{n=0}^{\infty}(n+1)x^n \\
&= \sum_{n=1}^{\infty} n(n+1)x^{n-1} \\
&= \sum_{n=0}^{\infty}(n+1)(n+2)x^n \\
&\downarrow \text{ multiply by } \frac{x}{2} \\
\frac{x}{(1-x)^3} &= \sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^{n+1} \\
&= 0 + \frac{1(2)}{2}x + \frac{2(3)}{2}x^2 + \frac{3(4)}{2}x^3 + \ldots \\
&= u_0 + u_1 x + u_2 x^2 + u_3 x^3 + \ldots
\end{aligned}$$

So, in the final analysis, the recurrence $u_0 = 0$ and $u_{n+1} = u_n + (n+1)$, which implies

- $u_0 = 0$
- $u_1 = 0 + 1 = 1$
- $u_2 = 1 + 2$
- $u_3 = 1 + 2 + 3$
- etc.

which results in the familiar formula for the 'Triangular Numbers'

$$u_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

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Homogeneous Linear Recurrences

An HLR is a sequence $\{u_n\}$ where $u_0 = c_0$, $u_1 = c_1, \ldots, u_{k-1} = c_{k-1}$ for constants $c_0, \ldots, c_{k-1}$ and

$$u_{n+k} + (a_1 u_{n+k-1} + \cdots + a_k u_n) = 0$$

for $n \geq 0$, namely that u_{n+k} is expressed as a linear combination of the k previous terms in the sequence.

For example, for the Fibonacci sequence $f_0 = 1$, $f_1 = 1$, (whence $k = 2$) and $f_{n+2} = f_{n+1} + f_n$, namely

$$f_{n+2} - 1f_{n+1} - 1f_n = 0$$

namely $a_1 = -1$ and $a_2 = -1$.

The usage of the term 'homogeneous' is due to the recurrence relation being expressed as $u_{n+k} + (a_1 u_{n+k-1} + \cdots + a_k u_n) = 0$, i.e. the right hand side being 0.

As such, the example earlier $u_0 = 1$, $u_n = 3u_{n-1} + 1$ is *not* homogeneous since $u_n - 3u_{n-1} = 1$.

You may say, what if we subtract 1 to get $u_n - 3u_{n-1} - 1 = 0$?

But the resulting equation on the left is not a linear combination of the sequence terms as it was for Fibonacci.

Inhomogeneous Recursion Relations

For example: $u_0 = 0,\ u_1 = 0,\ u_{n+2} = 3u_{n+1} + 4u_n + (n+3)$ for $n \geq 2$.

Here we define:

$$
\begin{aligned}
U(x) &= u_0 + u_1 x + u_2 x^2 + u_3 x^3 + \dots \\
&= u_0 + u_1 x + (3u_1 + 4u_0 + 3)x^2 + (3u_2 + 4u_1 + 4)x^3 + \dots \\
&= u_0 + u_1 x + (3u_1 x^2 + 3u_2 x^3 + \dots) + (4u_0 x^2 + 4u_1 x^3 + \dots) + (3x^2 + 4x^3 + 5 \\
&\downarrow\ 3xU(x) = 3u_0 x + 3u_1 x^2 + \dots \\
&\downarrow\ 4x^2 U(x) = 4u_0 x^2 + 4u_1 x^3 + \dots \\
&= 3xU(x) + 4x^2 U(x) + (3x^2 + 4x^3 + \dots) \\
&= 3xU(x) + 4x^2 U(x) + \left[\frac{1}{(1-x)^2} - (1+2x)\right]
\end{aligned}
$$

So we have that

$$U(x)(1-3x-4x^2) = \frac{1}{(1-x)^2} - (1+2x) = \frac{3x^2-2x^3}{(1-x)^2}$$

and so we solve for $U(x)$ to get

$$\begin{aligned} U(x) &= \frac{x^2(3-2x)}{(1-4x)(1+x)(1-x)^2} \\ &= \frac{A}{1-4x} + \frac{B}{1+x} + \frac{C}{1-x} + \frac{D}{(1-x)^2} \end{aligned}$$

$$\frac{A}{1-4x} + \frac{B}{1+x} + \frac{C}{1-x} + \frac{D}{(1-x)^2}$$

$$= \frac{2}{9}\left(\frac{1}{1-4x}\right) + \frac{1}{4}\left(\frac{1}{1+x}\right) - \frac{11}{36}\left(\frac{1}{1-3x}\right) - \frac{1}{6}\left(\frac{1}{(1-x)^2}\right)$$

To complete the analysis we use the following series representations:

$$\frac{1}{1-4x} = \sum_{n=0}^{\infty}(4x)^n = \sum_{n=0}^{\infty} 4^n x^n$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n$$

This implies that

$$U(x) = \sum_{n=0}^{\infty} \underbrace{\left[\left(\frac{2}{9}\right) 4^n + \left(\frac{1}{4}\right) (-1)^n - \left(\frac{11}{36}\right) 1^n - \left(\frac{1}{6}(n+1)\right) \right]}_{u_n} x^n$$

which is not something one would 'guess' or be easily able to determine from looking at the first few terms.

Here is a slightly different example, but still uses many of the same techniques.

Recall the triangular numbers formula

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

for each integer $n \geq 1$.

This is usually proven by induction, but as we saw, this can be done with generating functions.

What if we want the analogous formula for squares, namely:

$$1^2 + 2^2 + \cdots + n^2 = f(n)$$

for some function f.

To handle this case, we will use utilize the well known fact that $\frac{d}{dx}x^n = nx^{n-1}$, which we used earlier to derive the fact that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\downarrow$$

$$\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$$

$$\downarrow$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$$

but we utilize the fact that the above method is true for finite versions of the sum on the right.

You may have seen this fact given in Calculus,

$$1 + x + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}$$

provided $x \neq 1$. We can manipulate both sides of this equation.

$$1 + x + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}$$

$$\downarrow$$

$$\frac{d}{dx}\left(1 + x + \cdots + x^{n-1}\right) = \frac{d}{dx}\left(\frac{x^n - 1}{x - 1}\right)$$

$$\downarrow$$

$$1 + 2x + 3x^2 + \cdots + (n-1)x^{n-2} = \frac{-x^{n-1}n + 1 + (n-1)\,x^n}{(x-1)^2}$$

$$\downarrow \text{ multiply by } x$$

$$x + 2x^2 + 3x^3 + \cdots + (n-1)x^{n-1} = \frac{-x^n n + x + (n-1)\,x^{n+1}}{(x-1)^2}$$

$$x + 2x^2 + 3x^3 + \cdots + (n-1)x^{n-1} = \frac{-x^n n + x + (n-1)\,x^{n+1}}{(x-1)^2}$$

$$\downarrow$$

$$\frac{d}{dx}\left(x + 2x^2 + 3x^3 + \cdots + (n-1)x^{n-1}\right) = \frac{d}{dx}\left(\frac{-x^n n + x + (n-1)\,x^{n+1}}{(x-1)^2}\right)$$

$$\downarrow$$

$$1 + 4x + 9x^2 + \cdots + (n-1)^2 x^{n-2} = \frac{((n-2)\,x - n)\,(x-1)\,nx^{n-1} + (x^n - 1)\,(x+1)}{(1-x)^3}$$

At this stage it looks very messy, but if we can 'let $x = 1$' on both sides, in particular the left we get

$$1 + 4 + 9 + \cdots + (n-1)^2 = 1^1 + 2^2 + 3^2 + \cdots + (n-1)^2$$

but letting $x = 1$ on the right yields the indeterminate form $\frac{0}{0}$.

So, even though we insisted that this theory does not involve calculus, we can make an exception in this case since it will very readily give us our answer.

Basically

$$1+4x+9x^2+\cdots+(n-1)^2x^{n-1}=\frac{((n-2)\,x-n)\,(x-1)\,nx^{n-1}+(x^n-1)\,(x+1)}{(1-x)^3}$$

$$\downarrow$$

$$\lim_{x\to 1} 1+4x+9x^2+\cdots+(n-1)^2x^{n-1}=\lim_{x\to 1}\frac{((n-2)\,x-n)\,(x-1)\,nx^{n-1}+(x^n-1)\,(x+1)}{(1-x)^3}$$

$$\downarrow \text{ L'Hopitals Rule}$$

$$1^2+2^2+\cdots+(n-1)^2=\frac{(n-1)n(2n-1)}{6}$$

which, implies

$$1^2+2^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

for each $n\geq 1$.