MA294 Lecture

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Partitions of a Positive Integer

For a positive integer n, a partition of n is a sum of positive integers that adds up to n.

$n = 1$

$$1 = 1$$

(no other possibilities)
$n = 2$

$$\begin{aligned} 2 &= 2 \\ &= 1 + 1 \end{aligned}$$

$n = 3$

$$\begin{aligned} 3 &= 3 \\ &= 2 + 1 \\ &= 1 + 1 + 1 \end{aligned}$$

Ok, not too difficult, but as n increases, the number of possibilities increases.

Consider $n = 5$

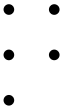
$$\begin{aligned} 5 &= 5 \\ &= 4+1 \\ &= 3+2 \\ &= 3+1+1 \\ &= 2+2+1 \\ &= 2+1+1+1 \\ &= 1+1+1+1+1 \end{aligned}$$

A way to understand these partitions, and to help see the 'symmetries' that exist amongst them is through what are known as Ferrers Diagrams.

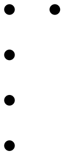
We've enumerate partitions by following a certain convention.

Namely, we write partitions typically as $n = i_1 + i_2 + \cdots + i_k$ where $i_1 \geq i_2 \geq \cdots \geq i_k$, such as $5 = 2 + 2 + 1$.

We can represent this as a Ferrers Diagram which is a 'stack' of dots arranged in rows of non-increasing size.



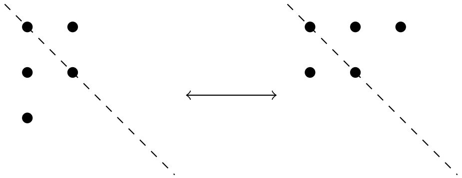
$5 = 2 + 2 + 1$



$5 = 2 + 1 + 1 + 1$

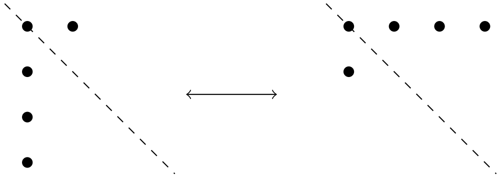
$5 = 1 + 1 + 1 + 1 + 1$

What is rather nice about these diagrams is that they can be 'flipped' to yield different partitions.



$5 = 2 + 2 + 1$ $\qquad$ $5 = 3 + 2$

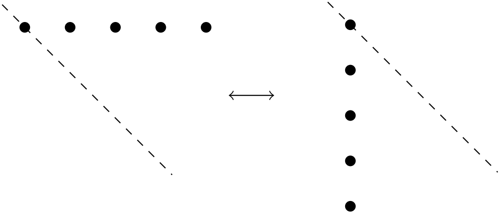
So we say these partitions are 'conjugate', and similarly



$5 = 2 + 1 + 1 + 1$ $\qquad$ $5 = 4 + 1$

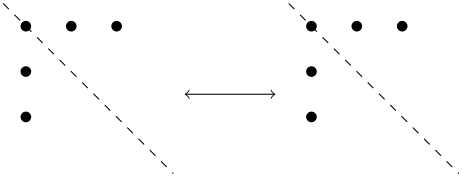
are conjugate as well.

Indeed all partitions can grouped in 'pairs' of partitions which are conjugates of each other, even these two:



$5 = 5 \qquad\qquad 5 = 1 + 1 + 1 + 1 + 1$

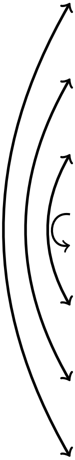
but some can be 'self conjugate', for example:



$5 = 3 + 1 + 1$ $\qquad$ $5 = 3 + 1 + 1$

where the 'self conjugacy' is due to the symmetry of the diagram when flipped.

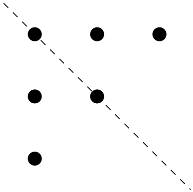
Applied to $n = 5$ we have the following correspondences:



$$
\begin{aligned}
5 &= 5 \\
&= 4 + 1 \\
&= 3 + 2 \\
&= 3 + 1 + 1 \\
&= 2 + 2 + 1 \\
&= 2 + 1 + 1 + 1 \\
&= 1 + 1 + 1 + 1 + 1
\end{aligned}
$$

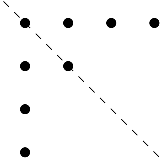
Applied to the $n = 6$ we have these correspondences:

$$
\begin{aligned}
6 &= 6 \\
&= 5 + 1 \\
&= 4 + 2 \\
&= 4 + 1 + 1 \\
&= 3 + 3 \\
&= 3 + 2 + 1 \\
&= 3 + 1 + 1 + 1 \\
&= 2 + 2 + 2 \\
&= 2 + 2 + 1 + 1 \\
&= 2 + 1 + 1 + 1 + 1 \\
&= 1 + 1 + 1 + 1 + 1 + 1
\end{aligned}
$$

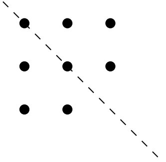


One may wonder if there are ever more than one self-conjugate partition of an integer.

Yes, consider $n = 8$.



$8 = 4 + 2 + 1 + 1$



$8 = 3 + 3 + 2$

In general, the number of self-conjugate partitions is a non-decreasing function of n.

And, speaking of numbers, how many partitions are there of an arbitrary integer n?

Definition

For $n \geq 0$ an integer, let $\mathcal{P}(n)$ be the number of partitions of n.

Consider:

- $\mathcal{P}(0) = 1$(i.e. $0 = 0$ only)
- $\mathcal{P}(1) = 1$
- $\mathcal{P}(2) = 2$
- $\mathcal{P}(3) = 3$
- $\mathcal{P}(4) = 5$
- $\mathcal{P}(5) = 7$
- $\mathcal{P}(6) = 11$

So the question is whether there is an obvious pattern, and also is $\mathcal{P}(n) \approx n$ as we see in these initial values?

We can give some indication of the answer to the second question by looking at $\mathcal{P}(n)$ for larger n.
$\mathcal{P}(10) = 42$

$$10 = 10$$
$$10 = 9 + 1$$
$$10 = 8 + 2$$
$$10 = 8 + 1 + 1$$
$$10 = 7 + 3$$
$$10 = 7 + 2 + 1$$
$$10 = 7 + 1 + 1 + 1$$
$$10 = 6 + 4$$
$$10 = 6 + 3 + 1$$
$$10 = 6 + 2 + 2$$
$$10 = 6 + 2 + 1 + 1$$
$$10 = 6 + 1 + 1 + 1 + 1$$
$$10 = 5 + 5$$
$$10 = 5 + 4 + 1$$
$$10 = 5 + 3 + 2$$
$$10 = 5 + 3 + 1 + 1$$
$$10 = 5 + 2 + 2 + 1$$
$$10 = 5 + 2 + 1 + 1 + 1$$
$$10 = 5 + 1 + 1 + 1 + 1 + 1$$
$$10 = 4 + 4 + 2$$
$$10 = 4 + 4 + 1 + 1$$

$10 = 4 + 3 + 3$

$10 = 4 + 3 + 2 + 1$

$10 = 4 + 3 + 1 + 1 + 1$

$10 = 4 + 2 + 2 + 2$

$10 = 4 + 2 + 2 + 1 + 1$

$10 = 4 + 2 + 1 + 1 + 1 + 1$

$10 = 4 + 1 + 1 + 1 + 1 + 1 + 1$

$10 = 3 + 3 + 3 + 1$

$10 = 3 + 3 + 2 + 2$

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$10 = 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1$

$10 = 2 + 2 + 2 + 2 + 2$

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$10 = 2 + 2 + 1 + 1 + 1 + 1 + 1 + 1$

$10 = 2 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$

$10 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$

- $\mathcal{P}(20) = 627$
- $\mathcal{P}(30) = 5604$
- $\mathcal{P}(40) = 37338$
- $\mathcal{P}(50) = 204226$
- $\mathcal{P}(60) = 966467$
- $\vdots$
- $\mathcal{P}(100) = 190,569,292$
- $\mathcal{P}(200) = 3,972,999,029,388$

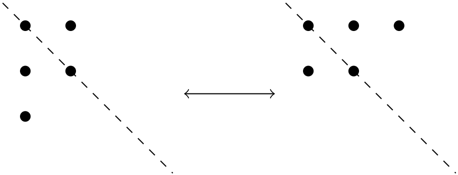
So you can see the value of $\mathcal{P}(n)$ grows quite dramatically.

Before we get back to the long term behavior of $\{\mathcal{P}(n)\}$ I wish to explore a bit more about partitions and their conjugates.

Notation: If λ is a partition of n then we sometimes write $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ where $n = \lambda_1 + \cdots + \lambda_k$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ which is the same, non-increasing pattern as before.

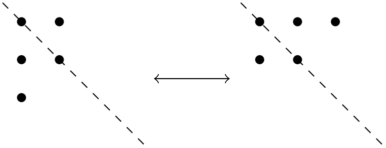
We will also use the term 'part' synonymously with 'summand' when speaking of those terms which add up to n.

And for a partition λ, the conjugate partition is denoted λ^*, e.g. for $n = 5$, if $\lambda = (2,2,1)$ then $\lambda^* = (3,2)$ as we saw from the Ferrer's diagram of the two:



$\lambda = (2,2,1)$ $\qquad\qquad$ $\lambda^* = (3,2)$

Moreover, the operation $\lambda \mapsto \lambda^*$ is an 'involution' namely that repeated twice we get back the original partition, that is $(\lambda^*)^* = \lambda$.



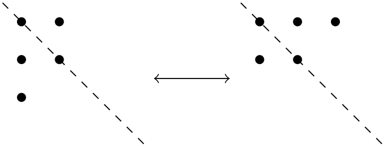
$\lambda = (2, 2, 1)$ $\qquad$ $\lambda^* = (3, 2)$

We used the Ferrers diagram to determine λ^* from λ but it actually can be done without the diagram.

Observe that in $\lambda = (\lambda_1, \ldots, \lambda_k)$ the largest part (summand) is λ_1, which, (visually) is the number of dots in the first row, which becomes the number of parts (i.e. the length) of λ^*.

Moreover, we also have that the first *column* in the Ferrers diagram for λ becomes the first row of λ^*.

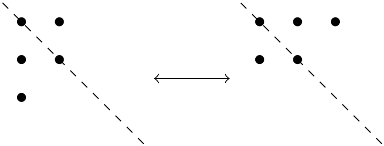
But the length of the first column of the diagram for λ is the largest part that appears in λ^*.



$\lambda = (2,2,1)$ $\qquad\qquad$ $\lambda^* = (3,2)$

i.e. If we conjugate $\lambda = (\lambda_1, \ldots, \lambda_k) \mapsto \lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_l^*)$ then the largest part (summand) of λ^* is exactly k.

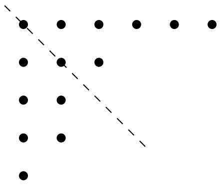
That is $\lambda_1^* = k$ and $\lambda_1 = l$.



$\lambda = (2, 2, 1)$ $\qquad$ $\lambda^* = (3, 2)$

Going further, λ^*_i is the length of column i in the diagram of λ and there is a dot in each position in this column, for every row of length at least i.

So λ^*_i is the number of parts of size $\geq i$ in λ.



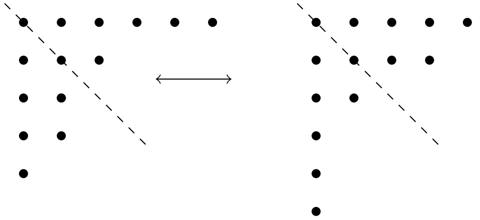
$\lambda = (6, 3, 2, 2, 1)$

So, for example if $\lambda = (6, 3, 2, 2, 1)$ (which has length 5) then $\lambda^* = (\lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*, \lambda_5^*, \lambda_6^*)$ where now $\lambda_1^* = 5$ (the length of λ).

$\lambda_2^* = 4$ since there are four parts of λ that are ≥ 2 (i.e. $6, 3, 2, 2$)

Similarly $\lambda_3^* = 2$ (accounting for $6, 3$), and $\lambda_4^* = 1$ (6 only), $\lambda_5^* = 1$, $\lambda_6^* = 1$ too, again accounting for the 6.

That is $\lambda^* = (5, 4, 2, 1, 1, 1)$, which we can confirm visually



$\lambda = (6, 3, 2, 2, 1)$ $\qquad$ $\lambda^* = (5, 4, 2, 1, 1, 1)$