MA294 Lecture

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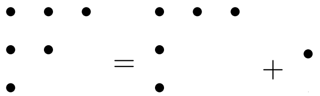
As to the self conjugate partitions, we have the following:

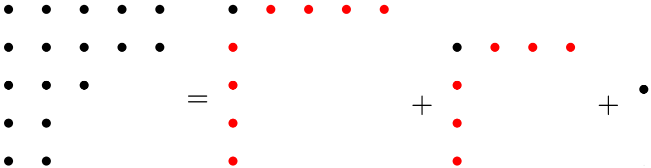
Theorem

The number of self-conjugate partitions of n is equal to the number of partitions of n into distinct, odd parts.

Proof We can give a bijection ψ : {self-conjugate partitions of n} $\to$ {partitions of n with distinct odd parts}. This is easily demonstrated in terms of the Ferrers diagram:



where we have $\psi((3,2,1)) = (5,1)$



where we have $\psi((5,5,3,2,2)) = (9,7,1)$ and the values 9, 7, 1 are all odd because they are 'hook' shaped, consisting of even numbers of dots on each side, meeting at a single point in the upper left, so an odd number overall.

The argument for an arbitrary self-conjugate partition is that ψ of each such partition is similarly a partition of n with odd parts corresponding to the sum of these 'hooks'.

This mapping is 1-1 since a different hook within a given self-conjugate partition yields a different part in ψ of that partition.

This mapping is also onto since if we have a partition of n consisting of odd parts $o_1, o_2, \ldots, o_k$ where $o_1 \geq o_2 \cdots \geq o_k$, then this partition can be put together into a self conjugate partition. (i.e. just reverse the process depicted in the diagrams above)

So consider the self-conjugate partitions of $n = 20$, which we can determine by finding the partitions of 20 consisting of distinct odd parts, i.e. $\psi(\mathit{self} - \mathit{conjugate})$.

What helps determine these is that n is even, so we look for an even number of distinct odd numbers which add up to 20.

- $(13, 7) = \psi(7, 5, 2, 2, 2, 1, 1)$
- $(17, 3) = \psi(9, 3, 2, 1, 1, 1, 1, 1, 1)$
- $(11, 5, 3, 1) = \psi(6, 4, 4, 4, 1, 1)$
- $(9, 7, 3, 1) = \psi(5, 5, 4, 4, 2)$
- $(11, 9) = \psi(6, 6, 2, 2, 2, 2)$
- $(19, 1) = \psi(10, 2, 1, 1, 1, 1, 1, 1, 1, 1)$
- $(15, 5) = \psi(8, 4, 2, 2, 1, 1, 1, 1)$

The interest in looking at partitions satisfying certain conditions leads to different variants of the function $\mathcal{P}(n)$.

Definition

For a positive integer n:

- $\mathcal{P}(n,k)$ is the number of partitions of n with k parts.
- $\mathcal{P}(n)$ is the number of partitions of n.
- $\mathcal{Q}(n,k)$ is the number of partition of n with k distinct parts.
- $\mathcal{Q}(n)$ is the number of partition of n with distinct parts.

- $\mathcal{P}(5,2) = 2$, since $5 = 4+1 = 3+2$,
- $\mathcal{P}(5,3) = 2$ since $5 = 2+2+1 = 3+1+1$,
- $\mathcal{Q}(5,1) = 1$ since $5 = 5$,
- $\mathcal{Q}(5,2) = 2$ since $5 = 4+1 = 3+2$.

The Behavior of $\mathcal{P}(n)$

If we consider the sequence $\{\mathcal{P}(n)\}_{n=0}^{\infty}$ we may ask whether there is an explicit formula for $\mathcal{P}(n)$.

Perhaps one may find a generating function for $\mathcal{P}(n)$.

If there were such a function $P(x)$ then it would have the form

$$P(x) = \mathcal{P}(0) + \mathcal{P}(1)x + \mathcal{P}(2)x^2 +$$

but the issue is, there is no recurrence relation that can be used to write $P(x)$ in terms of an equation involving itself.

Before we tackle this general problem, we can consider this 'simpler' problem which utilizes some ideas we will explore in more generality later.

Indeed, you may have seen this technique if you've had a precursor discrete mathematics class, as this is a kind of standard technique in combinatorics.

Problem: Find the number of solutions of $i + j + k = 10$ where i, j, and k are integers subject to the following constraints:

$$2 \leq i \leq 5$$
$$3 \leq j \leq 6$$
$$4 \leq k \leq 8$$

The condition '$i + j + k = 10$' can be viewed in terms of exponents of a variable, say 'x' in polynomial expression.

Specifically, I assert that the answer to this problem is to be found by computing the coefficient of x^{10} in the product

$$(x^2 + x^3 + x^4 + x^5) \cdot (x^3 + x^4 + x^5 + x^6) \cdot (x^4 + x^5 + x^6 + x^7 + x^8)$$

which makes sense in that we are looking for all products $x^i \cdot x^j \cdot x^k = x^{i+j+k}$ where $i + j + k = 10$.

That is, we view each parenthesized factor as corresponding to 'i', 'j', and 'k'.

i.e.

$$\underbrace{(x^2+x^3+x^4+x^5)}_{i}\cdot\underbrace{(x^3+x^4+x^5+x^6)}_{j}\cdot\underbrace{(x^4+x^5+x^6+x^7+x^8)}_{k}$$

The product overall is

$$x^9+3\,x^{10}+6\,x^{11}+10\,x^{12}+13\,x^{13}+14\,x^{14}+13\,x^{15}+10\,x^{16}+6\,x^{17}+3\,x^{18}+x^{19}$$

and so the answer is 3.

And it also shows the number of ways of adding these summands to achieve '11' for example, namely the coefficient of x^{11} which is '6'.

Now this is a partition problem in that we are partitioning 10 into positive integers, but here we are 'constraining' the possible summands.

$$(x^2 + x^3 + x^4 + x^5) \cdot (x^3 + x^4 + x^5 + x^6) \cdot (x^4 + x^5 + x^6 + x^7 + x^8)$$

- $2 + 3 + 5 = 10$
- $2 + 4 + 4 = 10$
- $3 + 3 + 4 = 10$

We'll see more examples covered in greater detail going forward.

For general partition questions, including those where there are no constraints on summands, we use series.

In particular, the strategy we employ is built upon our old friend the geometric series.

Consider

$$(1-x^i)^{-1} = 1 + x^i + x^{2i} + x^{3i} + \dots$$

where the coefficient of $x^{\alpha i}$ being 1 represents the number of partitions of $n = \alpha i$ where each summand is i, i.e.

$$\underbrace{i + i + \cdots + i}_{\alpha}$$

And of course, if $n \neq \alpha i$ for any α implies that the coefficient is zero since it's impossible!

So what we find is that $(1-x^i)^{-1}$ is the generating function for the sequence $\{f_n\}$ where

$$f_n = f(n) = \begin{cases} 1 \text{ if } n = \alpha i \\ 0 \text{ if } n \neq \alpha i \end{cases}$$

since

$$(1-x^i)^{-1} = 1 + x^i + x^{2i} + x^{3i} + \dots$$

Consider now the count of those partitions where the summands are 2 or 3, for example:

$$
\begin{aligned}
10 &= 3+3+2+2 \\
&= 2+2+2+2+2 \\
12 &= 3+3+3+3 \\
&= 3+3+2+2+2 \\
&= 2+2+2+2+2+2
\end{aligned}
$$

which are not, for the given integers on the left, the full set of partitions of those integers.

How do we enumerate these partitions?

Consider $(1-x^2)^{-1}(1-x^3)^{-1}$ which are each series, which when multiplied together yield:

$$(1+x^2+x^4+x^6+x^8+x^{10}+x^{12}+\dots)(1+x^3+x^6+x^9+x^{12}+\dots)$$
$$=(1+x^2+x^3+x^4+x^5+2x^6+x^7+2x^8+2x^9+2x^{10}+2x^{11}+\boxed{3}x^{12}+.$$

For example, the coefficient of $\boxed{3}$ in front of x^{12} arises from the products $x^{12}\cdot 1$, $x^6\cdot x^6$, and $1\cdot x^{12}$, which, in turn, correspond to partitions, i.e.

- $x^{12}\cdot 1 \leftrightarrow (2+2+2+2+2+2)$
- $x^6\cdot x^6 \leftrightarrow (2+2+2)+(3+3)$
- $1\cdot x^{12} \leftrightarrow (3+3+3+3)$

So for two arbitrary summands, in the product of the series $(1-x^i)^{-1}(1-x^j)^{-1}$ the coefficient of x^n is contributed to by any partition of n of the form $\alpha i + \beta j$.

In a similar fashion, if we extend to three different allowable summands, i, j, and k, we consider the series product

$$(1-x^i)^{-1}(1-x^j)^{-1}(1-x^k)^{-1}$$

where those x^n with non-zero coefficient correspond to those n expressible as $n = \alpha i + \beta j + \gamma k$.

Here is a neat application of this idea.

At a farm stand, plums cost 20¢, peaches cost 30¢, oranges cost 40¢, and grapefruits cost 50¢.

How many different fruit combinations can be purchased with $1.20?

Initial Answer: Find the coefficient of 120 in the generating function

$$(1 - x^{20})^{-1}(1 - x^{30})^{-1}(1 - x^{40})^{-1}(1 - x^{50})^{-1}$$

which, while correct, can be simplified by realizing that every amount in this problem is a multiple of 10 (¢), including the $1.20 which is 120¢.

However, these powers are rather large, but we can re-frame the problem in order not to have to work with such large numbers.

Instead, let's determine the coefficient of x^{12} in $(1-x^2)^{-1}(1-x^3)^{-1}(1-x^4)^{-1}(1-x^5)^{-1}$.

$$
\begin{aligned}
&(1+x^2+x^4+x^6+x^8+x^{10}+x^{12}+\dots)\cdot \\
&(1+x^3+x^6+x^9+x^{12}+\dots)\cdot \\
&(1+x^4+x^8+x^{12}+\dots)\cdot \\
&(1+x^5+x^{10}+\dots)
\end{aligned}
$$

which is precisely $\boxed{10}x^{12}$.

Why? Think $(x^{2i})(x^{3j})(x^{4k})(x^{5l})$=(plum)(peach)(orange)(grapefruit)

- $x^{12} \cdot 1 \cdot 1 \cdot 1$ {6 plum}
- $x^{8} \cdot 1 \cdot x^{4} \cdot 1$ {4 plum, 1 orange}
- $x^{6} \cdot x^{6} \cdot 1 \cdot 1$ {3 plum,2 peach}
- $x^{4} \cdot 1 \cdot x^{8} \cdot 1${2 plum,2 orange }
- $x^{4} \cdot x^{3} \cdot 1 \cdot x^{5}${2 plum,1 orange,1 grapefruit}
- $x^{2} \cdot x^{6} \cdot x^{4} \cdot 1${1 plum,2 peaches,1 orange}
- $x^{2} \cdot 1 \cdot 1 \cdot x^{10}${1 plum,2 grapefruit}
- $1 \cdot x^{12} \cdot 1 \cdot 1${4 peaches}
- $1 \cdot 1 \cdot x^{12} \cdot 1${3 orange}
- $1 \cdot x^{3} \cdot x^{4} \cdot x^{5}${1 peach, 1 orange,1 grapefruit}