MA294 Lecture

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Going beyond finite lists of specific parts (summands), we consider what happens if we include all possible values for parts.

Theorem

The generating function for the sequence $\{\mathcal{P}(n)\}_{n=0}^{\infty}$ *is*

$$\begin{aligned} P(x) &= \prod_{i=1}^{\infty}(1-x^i)^{-1} \\ &= (1-x^1)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\cdots \end{aligned}$$

in that $P(x) = \sum_{n=0}^{\infty}\mathcal{P}(n)x^n$.

The degree n coefficients in $(1-x^1)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\cdots$ correspond to those products $x^{i_1}x^{i_2}\cdots x^{i_k}$ where $i_1+i_2\cdots+i_k=n$ and since all powers x^{i_t} appear in at least once in

$$(1-x^i)^{-1}=\frac{1}{1-x^i}=1+x^i+x^{2i}+\dots$$

then all possible partitions of n occur as powers of x in the 'infinite product' $\prod_{i=1}^{\infty}(1-x^i)^{-1}$.

Theorem

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$$
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&= (1-x^1)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\cdots
\end{aligned}
$$

in that $P(x) = \sum_{n=0}^{\infty}\mathcal{P}(n)x^n$.

To understand this theorem, we realize that $(1-x)^{-1}$ contributes 1 summand, $(1-x^2)^{-1}$ contributes 2 summands, etc. and so for each n, we have contributions from all $x^{\alpha i}$ corresponding to those partitions of n containing α instances of the summand 'i'.

However, we may consider more nuanced counting questions.

Consider the partitions of all integers less than some N by restricting to the product

$$(1-x)^{-1}(1-x^2)^{-1}\cdots(1-x^N)^{-1}$$

as no partition of N can contain summands larger than N.

For example, suppose $N = 5$, then we can consider

$$(1-x^1)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}(1-x^4)^{-1}(1-x^5)^{-1}$$

but, in fact, we only need those terms of degree below 5.

$$(1+x+x^2+x^3+x^4+x^5)(1+x^2+x^4)(1+x^3)(1+x^4)(1+x^5) =$$
$$1+x+2x^2+3x^3+5x^4+7x^5+\cdots+x^{21}$$

and we read off the exact values

$\mathcal{P}(0) = 1, \mathcal{P}(1) = 1, \mathcal{P}(2) = 2, \mathcal{P}(3) = 3, \mathcal{P}(4) = 5$, and $\mathcal{P}(5) = 7$.

If one adds more terms, one obtains $\mathcal{P}(n)$ for still larger n.

Before going further, let's make note of a remarkable fact due to Hardy and Ramanujan about the magnitude of $\mathcal{P}(n)$.

What they showed was this

$$\mathcal{P}(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$$

as $n \to \infty$ which is what is termed an asymptotic formula in that it gets better as n increases which means that

$$\lim_{n\to\infty} \frac{\mathcal{P}(n)}{\left(\frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}\right)} = 1$$

And speaking of (Srinivasa) Ramanujan, he also discovered a number extraordinary 'modular' properties of $\mathcal{P}(n)$.

Theorem

The following are true:

$$\mathcal{P}(5k+4) \equiv 0 \ (mod\ 5)$$
$$\mathcal{P}(7k+5) \equiv 0 \ (mod\ 7)$$
$$\mathcal{P}(11k+6) \equiv 0 \ (mod\ 11)$$

For example,

- $\mathcal{P}(4) = \mathcal{P}(5 \cdot 0 + 4) = 5$
- $\mathcal{P}(9) = \mathcal{P}(5 \cdot 1 + 4) = 15$
- $\mathcal{P}(5) = \mathcal{P}(7 \cdot 0 + 5) = 7$
- $\mathcal{P}(12) = \mathcal{P}(7 \cdot 1 + 5) = 77$

but it makes one wonder if such congruences hold true for other moduli?

Many years later, this similar (but more complicated) congruence relation was discovered:

$$\mathcal{P}(11^3 \cdot 13k + 237) \equiv 0 \ (mod \ 13)$$

and there are some deeper results which are outside the scope of this class.

A good many of these ideas, in particular that $P(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$ is the generating function for $\mathcal{P}(n)$, originated in the work of Euler (Think $e^{\pi i} + 1 = 0$) in the 1700's.

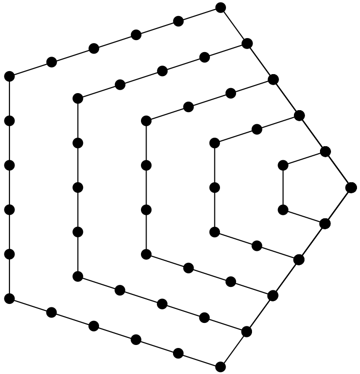
Here is another amazing result of Euler's which (like the result about $P(x)$, and even the proof that $e^{\pi i} + 1 = 0$) is based on series and infinite products.

Theorem

(Pentagonal Numbers Theorem)

$$\prod_{i=1}^{\infty}(1-x^i) = \sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{1}{2}n(3n-1)} = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \ldots$$

Here is the origin of the 'Pentagonal Numbers' sequence we just mentioned.



which has the terms $1, 5, 12, 22, 35, 51, \ldots$, and in general $p_n = \frac{3n^2 - n}{2}$.

This has a fundamental connection to $\mathcal{P}(n)$.

First, we must establish the fact that

$$\prod_{i=1}^{\infty}(1 - x^i) = \sum_{n=0}^{\infty}(\mathcal{P}_E(n) - \mathcal{P}_O(n))x^n$$

where $\mathcal{P}_E(n)$ and $\mathcal{P}_O(n)$ are the ways of partitioning n into either a sum of even or odd number of distinct parts.

The other fact we claim (which we won't prove here) is that

Lemma

$$\mathcal{P}_E(n) - \mathcal{P}_O(n) = (-1)^k$$

if $n = \frac{1}{2}k(3k+1)$ *for some* $k \in \mathbb{Z}$ *and* 0 *otherwise.*

So, using the Pentagonal Numbers Theorem, and the facts we just observed:

$$
\begin{aligned}
1 &= \frac{\prod_{i=1}^{\infty}(1-x^i)}{\prod_{i=1}^{\infty}(1-x^i)} \\
&= (1 - x - x^2 + x^5 + x^7 - \dots)(\mathcal{P}(0) + \mathcal{P}(1)x + \mathcal{P}(2)x^2 + \dots)
\end{aligned}
$$

which implies that

$$0 = \mathcal{P}(n) - \mathcal{P}(n-1) - \mathcal{P}(n-2) + \mathcal{P}(n-5) + \mathcal{P}(n-7) - \dots$$

which can be re-arranged to get a recurrence relation for $\mathcal{P}(n)$, namely

$$\mathcal{P}(n) = \mathcal{P}(n-1) + \mathcal{P}(n-2) - \mathcal{P}(n-5) - \mathcal{P}(n-7) + \mathcal{P}(n-12) + \dots$$

which actually is *not* an infinite sum, since $n \geq 0$, so $n - \boxed{?}$ is eventually < 0 so $\mathcal{P}(n - \boxed{?})$ is 0.

For example:

- $\mathcal{P}(6) = \mathcal{P}(5) + \mathcal{P}(4) - \mathcal{P}(1) = 7 + 5 - 1 = 11$
- $\mathcal{P}(7) = \mathcal{P}(6) + \mathcal{P}(5) - \mathcal{P}(2) - \mathcal{P}(0) = 11 + 7 - 2 - 1 = 15$
- $\mathcal{P}(8) = \mathcal{P}(7) + \mathcal{P}(6) - \mathcal{P}(3) - \mathcal{P}(1) = 15 + 11 - 3 - 1 = 22$
- $\mathcal{P}(9) = \mathcal{P}(8) + \mathcal{P}(7) - \mathcal{P}(4) - \mathcal{P}(2) = 22 + 15 - 5 - 2 = 30$
- $\mathcal{P}(10) = \mathcal{P}(9) + \mathcal{P}(8) - \mathcal{P}(5) - \mathcal{P}(3) = 30 + 22 - 7 - 3 = 42$