Turn in #6

Let E be the splitting field of $x^5 - 7 \in \mathbb{Q}[x]$.

We know that the roots of $x^5 - 7$ are of the form $\zeta^i \sqrt[5]{7}$
where ζ is a primitive fifth root of unity.

(a) Determine $[E : \mathbb{Q}]$.

(b) Find six subfields $K_i \subseteq E$, $i = 1, \dots, 6$ such that $[K_i : \mathbb{Q}] = 5$
for $i = 1, 2, 3, 4, 5$ and $[K_6 : \mathbb{Q}] = 4$

(c) Which of the K_i above are splitting fields over $\mathbb{Q}$ and which aren't?

(d) If $\tau(\zeta) = \zeta^2$, $\tau(\sqrt[5]{7}) = \sqrt[5]{7}$ and $\sigma(\sqrt[5]{7}) = \zeta\sqrt[5]{7}$, $\sigma(\zeta) = \zeta$
show that $G = Gal(E/\mathbb{Q})$ is generated by σ and τ and list the elements of G.
[Hint: For (d) determine $\tau\sigma\tau^{-1}$.]

[20 points]