Problem Set

1. Find the value of $\phi(20)$.
2. Find the value of $\phi(60)$.
3. Find the value of $\phi(63)$.
4. Find the value of $\phi(97)$.
5. Find the value of $\phi(341)$.
6. Find the value of $\phi(561)$.
7. Find the value of $\phi(8800)$.
8. Show that if n is odd, then $\phi(2n) = \phi(n)$.
9. Show that if n is even, then $\phi(2n) = 2\phi(n)$.
10. Let x be the *smallest* positive integer such that

$$2^x \equiv 1 \mod 63.$$

Find x, and verify that $x \mid \phi(63)$.

11. Find the last three digits of 7^{9999}.
12. (a) The positive divisors of 6 are 1, 2, 3, and 6. Compute $\phi(1) + \phi(2) + \phi(3) + \phi(6)$.

    (b) The positive divisors of 8 are 1, 2, 4, and 8. Compute $\phi(1) + \phi(2) + \phi(4) + \phi(8)$.

    (c) The positive divisors of 9 are 1, 3, and 9. Compute $\phi(1) + \phi(3) + \phi(9)$.

    (d) Let $n \geq 1$. Make a conjecture about the value of

$$\sum_{d|n} \phi(d),$$

where the sum is taken over all of the divisors d of n. Try to prove that your conjecture is correct. To prove that your conjecture is correct, it may be useful to use the result of the next problem.

13. Recall that a function f is said to be *multiplicative* if $f(mn) = f(m)f(n)$ for all integers m and n such that $(m, n) = 1$. We know that Euler's ϕ-function is multiplicative. Suppose that f is a multiplicative function, and define a new function $g(n)$ as follows:

$$g(n) = f(d_1) + f(d_2) + \cdots + f(d_r),$$

where $d_1, d_2, \ldots, d_r$ are the divisors of n. Show that $g(n)$ is multiplicative.

14. What can you say about n if the value of $\phi(n)$ is a prime number? What if the value of $\phi(n)$ is the square of a prime number.

15. Find at least five different numbers n such that $\phi(n) = 160$.

16. Suppose that the integer n satisfies $\phi(n) = 1000$. Make a list of all the primes that might possibly divide n. Use this information to find all integers n that satisfy $\phi(n) = 1000$.

17. Find all values of n that satisfy each of the following equations:
    (a) $\phi(n) = n/2$
    (b) $\phi(n) = n/3$
    (c) $\phi(n) = n/6$

18. Find the remainder of
$$10^{10} + 10^{10^2} + \cdots + 10^{10^{10}}$$
upon division by 7.

19. (a) For each integer $2 \le a \le 10$, find the last four digits of a^{1000}.
    (b) Based on your experiments in (a), and further experiments if necessary, give a simple criterion that allows you to predict the last four digits of a^{1000} from the value of a.
    (c) Prove that your criterion in (b) is correct.

20. Show that for all natural numbers s, there is an integer n divisible by s such that the sum of the digits of n is equal to s.

21. Prove that $504 \mid (n^9 - n^3)$ for all integers $n \ge 1$.

22. Prove that for any odd integer $n > 0$, $n \mid (2^{n!} - 1)$.

23. Prove that for every natural number n there exists some power of 2 whose final n digits are all ones and twos.

24. Prove that there exists a positive integer k such that $k \cdot 2^n + 1$ is composite for every positive integer n.

25. Suppose that p and q are different odd primes and that a is an integer such that $(a, pq) = 1$. Show that
$$a^{\phi(pq)/2} \equiv 1 \mod pq.$$

26. Show that if $n > 2$, then $2 \mid \phi(n)$.

27. In this series of exercises, you will prove the Euler-Fermat Theorem. Let n be an integer greater than or equal to 1, and let a be an integer such that $(a, n) = 1$. Let
$$1 = b_1 < b_2 < \cdots < b_{\phi(n)} < n$$
be the $\phi(n)$ numbers between 0 and n that are relatively prime to n.