

Ch.2 #17. A group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all a, b in G .

Proof. First, let's compute $(ab)^{-1}$ without any assumptions, i.e. just using the basic properties of a group. Basically we want to find x in G such that

$$x(ab) = e$$

So we need to solve for x . To do so we proceed as follows.

$$xab = e$$

$$xab \cdot b^{-1} = e \cdot b^{-1}$$

$$xa = b^{-1}$$

$$xa \cdot a^{-1} = b^{-1}a^{-1}$$

$$x = b^{-1}a^{-1}$$

Therefore $(ab)^{-1} = b^{-1}a^{-1}$.

If G is Abelian then certainly $(ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1}$.

→

Now, if we *assume* $(ab)^{-1} = a^{-1}b^{-1}$ then we have

$$b^{-1}a^{-1} = a^{-1}b^{-1}$$

and so we can argue as follows:

$$b^{-1}a^{-1} = a^{-1}b^{-1}$$

$$b \cdot b^{-1}a^{-1} = b \cdot a^{-1}b^{-1}$$

$$a^{-1} = ba^{-1}b^{-1}$$

$$a \cdot a^{-1} = a \cdot ba^{-1}b^{-1}$$

$$e = aba^{-1}b^{-1}$$

$$e \cdot b = aba^{-1}b^{-1} \cdot b$$

$$b = aba^{-1}$$

$$b \cdot a = aba^{-1} \cdot a$$

$$ba = ab$$

Hence G is Abelian. \square