Show that $Z(S_n) = \{i\}$ for $n \geq 3$

Recall that $Z(G) = \{x \in G | gx = xg \ \forall g \in G\}$. As such, to show that $Z(S_n)$ is trivial, it suffices to show that for all $\sigma \neq i \in S_n$, there exists $\tau \in S_n$ such that $\sigma\tau \neq \tau\sigma$.

This being the case, can we *easily* find a τ for every σ? The answer is yes, keeping in mind the fact that for δ, γ in S_n, $\delta = \gamma$ if and only if $\delta(x) = \gamma(x)$ for all $x \in X$. Therefore, using the reverse of this:

$$\sigma\tau \neq \tau\sigma \leftrightarrow \exists x \in X \ s.t. \ \sigma\tau(x) \neq \tau\sigma(x)$$

So let $\sigma \in S_n$ be a non identity element and say $\sigma(a) = b$ where $a \neq b$ then we can choose $\tau = (bc)$ where $c \neq \sigma(b)$, which is possible since $n \geq 3$.

$$\tau\sigma(a) = \tau(b) = c$$

while

$$\sigma\tau(a) = \sigma(a) = b$$

and since $c \neq \sigma(b)$ then we are done, because we've demonstrated that $\sigma\tau$ and $\tau\sigma$ act differently on $a \in X$ and so are different elements of S_n. As such, no nonidentity element of S_n commutes with all of S_n and so $Z(S_n) = \{i\}$.