Ch. 7 #25

$|G| = 33$

By LaGrange's Theorem, if $a \in G$ then $|a| \mid |G|$.
Thus $|a| = 1, 3, 11$ or 33.

If $G = \langle a \rangle$ then $|a^{11}| = 3$ so $b = a^{11}$ is our order 3 element.

If, however, every element of G (other than e) had order 11 then this would give 32 elements of order 11.

However, by the corollary to Theorem 4.4 we must have then

$$\phi(11) \mid 32$$

but $\phi(11) = 10$ and so we have a contradiction!

Ch. 7 # 22

Case I: G infinite

Pick $a \in G$, $a \neq e$, then either $\langle a \rangle$ is a proper subgroup of G already or $\langle a \rangle = G$. But if $\langle a \rangle = G$ then $H = \langle a^2 \rangle$ is then a proper subgroup of G.

(Consider $2\mathbb{Z} \leq \mathbb{Z}$)

Case II: G finite

Pick $a \in G$, $a \neq e$ then either $\langle a \rangle$ is a proper subgroup of G or $\langle a \rangle = G$.

If $\langle a \rangle = G$ then consider $\langle a^k \rangle$ for $k = 1, 2, \ldots, n-1$ where $n = |\langle a \rangle| = |G|$. If $\langle a^k \rangle = G$ for each such k then

$$|\langle a^k \rangle| = \frac{n}{\gcd(n,k)} = n \quad \text{for } k = 1, \ldots, n-1$$

$$\Downarrow$$

$$\gcd(n,k) = 1 \text{ for each } k = 1, \ldots, n-1$$

which implies that n is prime!