$$\frac{d}{dx} sin(x) = cos(x)$$

Given the graph of the unit circle below, with the labelled points $P(\cos(\theta), \sin(\theta))$, $Q(\cos(\theta), 0)$, $R(1, \tan(\theta))$, $S(1, 0)$ we notice the following.

$$\overline{RS} > \widehat{PS} > \overline{PQ}$$

where $\overline{RS}$ is the length of the line segment RS, $\widehat{PS}$ is the length of the arc PS and $\overline{PQ}$ is the length of the line segment PQ. Therefore

$$\tan(\theta) > \theta > \sin(\theta)$$

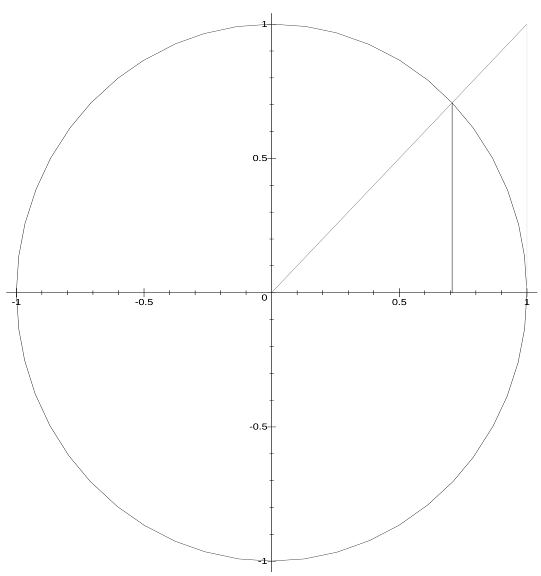
and since $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ this becomes

$$\frac{\sin(\theta)}{\cos(\theta)} > \theta > \sin(\theta)$$

Now if we take the reciprocal of each term we get

$$\frac{\cos(\theta)}{\sin(\theta)} < \frac{1}{\theta} < \frac{1}{\sin(\theta)}$$

By the sandwich theorem $\lim_{\theta \to 0^+} \frac{\sin(\theta)}{\theta} = 1.$



Now,

$$\begin{aligned}
\frac{d}{dx}sin(x) &= \lim_{h\to 0}\frac{sin(x+h)-sin(x)}{h}\\
&= \lim_{h\to 0}\frac{sin(x)cos(h)+cos(x)sin(h)-sin(x)}{h}\\
&= \lim_{h\to 0}\frac{sin(x)(cos(h)-1)+sin(h)cos(x)}{h}\\
&= \lim_{h\to 0}\frac{sin(x)(cos(h)-1)}{h}+\lim_{h\to 0}\frac{sin(h)cos(x)}{h}\\
&= sin(x)\lim_{h\to 0}\frac{cos(h)-1}{h}+cos(x)\lim_{h\to 0}\frac{sin(h)}{h}\\
&= sin(x)\cdot 0+cos(x)\cdot 1\\
&= cos(x)
\end{aligned}$$

As to the expression $\lim_{h\to 0}\frac{cos(h)-1}{h}=0$, we actually prove that $\lim_{h\to 0}\frac{1-cos(h)}{h}=0$

$$\begin{aligned}
\lim_{h\to 0}\frac{1-cos(h)}{h} &= \lim_{h\to 0}\frac{1-cos(h)}{h}\frac{1+cos(h)}{1+cos(h)}\\
&= \lim_{h\to 0}\frac{1-cos^2(h)}{h(1+cos(h))}\\
&= \lim_{h\to 0}\frac{sin^2(h)}{h(1+cos(h))}\\
&= \lim_{h\to 0}\frac{sin(h)}{h}\cdot\lim_{h\to 0}\frac{sin(h)}{1+cos(h)}\\
&= 1\cdot 0
\end{aligned}$$

since $\lim_{h\to 0}\frac{sin(h)}{1+cos(h)}=\frac{0}{1}=0$ and $\lim_{h\to 0}\frac{sin(h)}{h}=1$ was shown earlier.