Applications of the Antiderivative

Suppose the marginal cost function for a production problem is

$$C'(x) = 0.3x^2 + 2x$$

Determine the possible cost functions which have this as their marginal cost.

↳ Compute $\int 0.3x^2 + 2x \, dx$

$$= 0.3\left(\frac{x^3}{3}\right) + 2\left(\frac{x^2}{2}\right) + K$$ ← instead of C here to avoid confusion!

$$= 0.1x^3 + x^2 + K$$

ie $\boxed{C(x) = 0.1x^3 + x^2 + K}$ ← NOT A SINGLE FUNCTION BUT A FAMILY OF FUNCTIONS

Suppose we know that the fixed costs are \$2000, what is the <u>exact</u> cost function?

Well fixed costs = $C(0)$ since they are fixed regardless of the production level

ie $C(0) = 0.1(0)^3 + (0)^2 + K = 2000$

ie $K = 2000$

This is true in general

> FACT: If $f'(x)$ is known and $f(0)$ is known then antidifferentiation can be used to reconstruct the original $f(x)$.

SLIGHT VARIATION

Ex: Find the equation of the curve that passes through $(2, 6)$ if $\frac{dy}{dx} = 3x^2$ at any point x.

> HW
> 6.1: 1-23 odd
> 31, 33, 35-59 odd
> 67, 69, 83, 93

$$\rightsquigarrow \quad y' = 3x^2$$

$$\downarrow$$

$$y = 3\left(\frac{x^3}{3}\right) + C$$

$$= x^3 + C$$

ie $y = x^3 + C$ for <u>some</u> C

but now since $(2,6)$ lies on the graph then

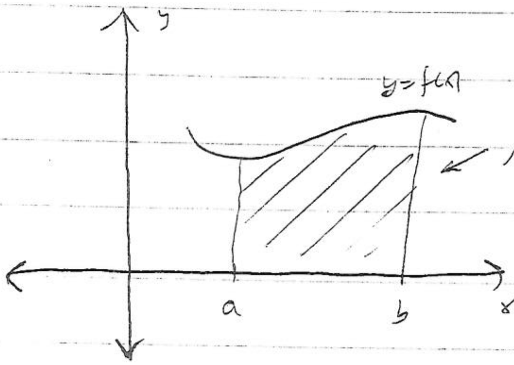
$$6 = (2)^3 + C$$

$$\to C = -2$$

ie $\boxed{y = x^3 - 2}$

The Definite Integral

Problem: Given the graph of a continuous function $f(x)$ on a closed interval $[a,b]$, determine the area underneath the graph b/w $x=a$ and $x=b$.

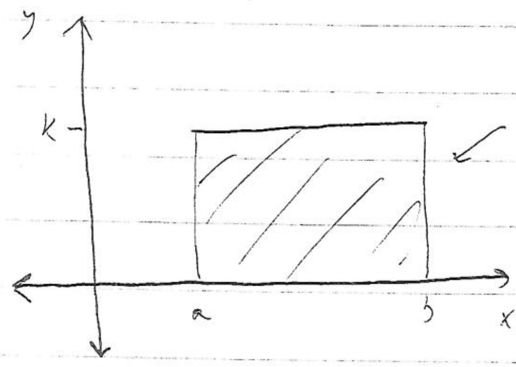


$A = ?$

(Assume $f(x) > 0$ on $[a,b]$)

Extremely Simple Example

$f(x) = k$ (constant) assume for now that $k > 0$



A = the area of a rectangle!

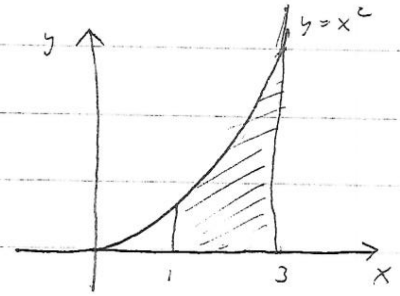
= height × width

$= k(b-a)$

This motivates the basic approach for computing these areas.

We can approximate the area under the curve by using rectangles

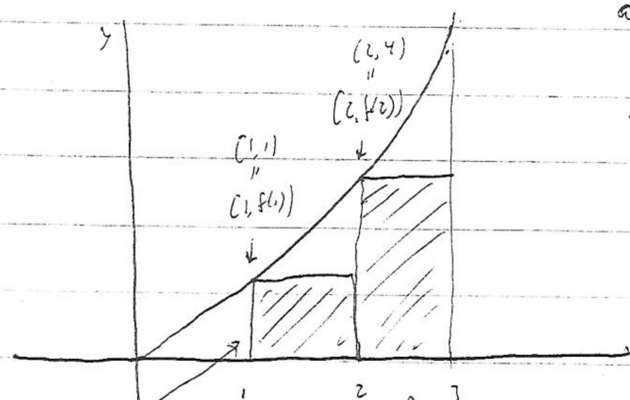
Ex: $f(x) = x^2$ on $[1,3]$



First attempt: Subdivide $[1,3]$ into two equal size 'subintervals' namely $[1,2]$ and $[2,3]$

Next, draw a rectangle over each subinterval whose height $= f$ (the left endpoint) and whose width is the length of the subinterval

ie



height $= f(1) = 1$
width $= 2 - 1 = 1$

$\downarrow$

area = height × width
$= 1 \cdot 1$
$= 1$

height $= f(2) = 4$
width $= 3 - 2 = 1$

$\downarrow$

area = height × width
$= 4 \cdot 1$
$= 4$

The point is that if A = area under the curve b/w $x=1$ and $x=3$ then

$$A \approx \text{sum of the areas of the rectangles}$$

$$= 1+4$$

$$= 5$$

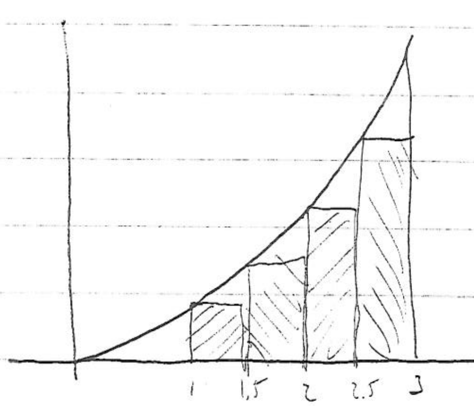
ie $A \approx 5$

(Of course this a crude estimate, so let's make it better!)

Subdivide $[1,3]$ into $n=4$ subintervals each of which will have length

$$\Delta x = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} = 0.5$$

namely $[1, 1.5]$, $[1.5, 2]$, $[2, 2.5]$, $[2.5, 3]$



$A \approx$ area of rectangle 1 + area of rectangle 2 + area of rectangle 3 + area of rectangle 4

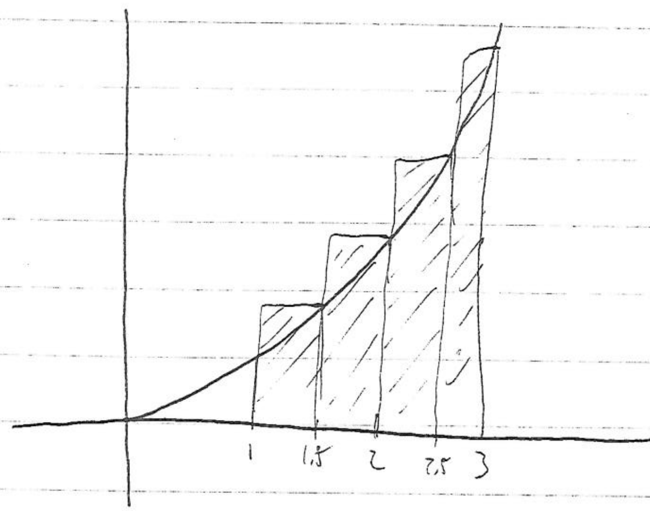
$$= f(1)(0.5) + f(1.5)(0.5) + f(2)(0.5) + f(2.5)(0.5)$$

$$= 1(0.5) + (2.25)(0.5) + (4)(0.5) + (6.25)(0.5) = 7$$

Is there some special reason for making the heights = f(left endpoints)

No

We could do the following



here

$$A \approx f(1.5)\cdot(0.5) + f(2)(0.5) + f(2.5)(0.5) + f(3)(0.5)$$

$$= (2.25)(0.5) + (4)(0.5) + (6.25)(0.5) + (9)(0.5)$$

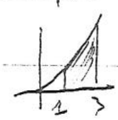
$$= 11$$

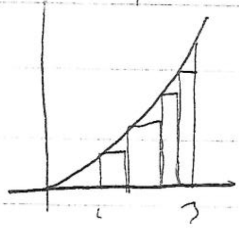
If we call this sum of 4 rectangles with heights determined by f(right endpoints) R_4 ~~then~~ and L_4 the previous case when the height = f(left endpoint) then

$$A \approx L_4 \quad \text{and} \quad A \approx R_4$$

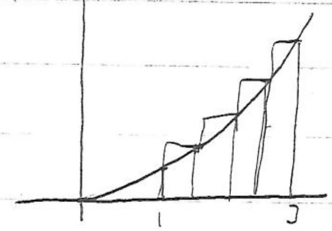
Note, by looking at the picture in each case, it's clear that (for $f(x) = x^2$ on $[1,3]$ at least) that

$$\begin{array}{ccccc} L_4 & \le & A & \le & R_4 \\ \| & & \uparrow & & \| \\ 7 & & & & 11 \end{array}$$





rectangles lie completely under the curve



rectangles cover over the curve

We can improve both approximations by increasing the number n of rectangles (which thereby decreases their width $\Delta x = \frac{b-a}{n}$!)

Ex: $L_8 = 7.6875$
$R_8 = 9.6875$ much closer to each other than for $n = 4$!

ie $L_8 < A < R_8$

FACT: If we define $L_n = \sum_{k=1}^{n} f(x_k^*)\Delta x$

x_k^* = left endpoint of each subinterval

$R_n = \sum_{k=1}^{n} f(x_k^*)\Delta x$

<u>right</u> endpoint of each interval

then it turns out that $\lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n = 8\frac{2}{3}$

In general if we subdivide $[a,b]$ into n equal subintervals of width $\Delta x = \frac{b-a}{n}$ and if $x_1^*, x_2^*, \ldots, x_n^*$ are points chosen within these subintervals (left endpoint, right endpoint or even midpoint) then ~~the~~ we can form the <u>Riemann sum</u>

$$\sum_{k=1}^{n} f(x_k^*)\Delta x \quad \text{(a sum of areas of rectangles)}$$

and observe that $A \approx \sum_{k=1}^{n} f(x_k^*)\Delta x$ and that this approximation gets better as n increases!

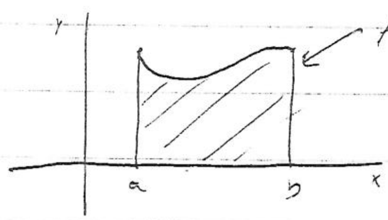
<u>Def</u>: The definite integral of $f(x)$ from a to b is defined as

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

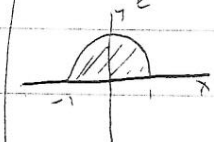
provided the limit exists.

<u>FACT</u>: If $f(x)$ is continuous on $[a,b]$ then $\int_a^b f(x)\,dx$ exists.

Note: If $f(x) > 0$ on $[a,b]$ then $\int_a^b f(x)\,dx$ is the area under the graph b/w $x=a$ and $x=b$



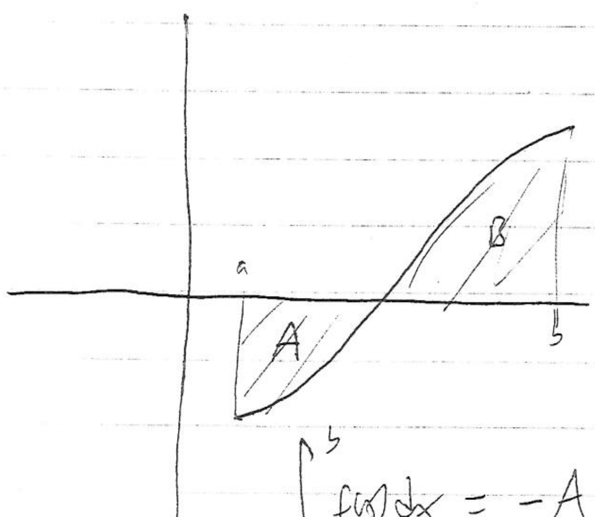
ex: $f(x) = \sqrt{1-x^2}$ on $[-1,1]$



$\int_{-1}^{1} \sqrt{1-x^2}\,dx$

$= \frac{1}{2}(\pi)$

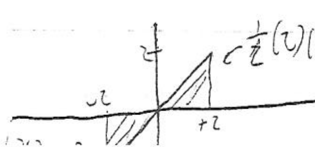
Note: If $f(x)$ is not positive throughout $[a,b]$ then $\int_a^b f(x)\,dx$ represents the 'net' area b/w the graph and the x-axis



A, B areas of the shaded regions

$$\int_a^b f(x)\,dx = -A + B$$

ex (slightly startling) $\int_{-2}^{2} x\,dx = 0$



not area = 0 !

Properties of Definite Integrals

$$\int_a^a f(x)\,dx = 0 \qquad \Leftarrow \Delta x = 0 \text{ for each } n!$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b k f(x)\,dx = k\left[\int_a^b f(x)\,dx\right]$$

HW 6.4

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

~~proof~~ without words!