MA532 Lecture

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Ordinal Numbers (continued)

We saw previously the following result:

Theorem (Law of Trichotomy for Ordinals)

Given any two ordinal numbers β and γ, exactly of one of the follow ing holds:

$$\beta < \gamma,\ \beta = \gamma,\ \beta > \gamma$$

In parallel to this we can prove the following result for cardinal numbers.

Theorem (Law of Trichotomy for Cardinals)

Given any two cardinal numbers b and a, exactly of one of the follow ing holds:

$$b < a,\ b = a,\ b > a$$

Proof.

The proof of this result is actually a quite straightforward application of the result we showed for ordinals.

The one fact that needs to be applied is that given sets A, B with certain cardinalities a, b, by the well ordering principle there exist orderings $\preceq_A$ and $\preceq_B$ where $(A, \preceq_A)$ and $(B, \preceq_B)$ are well ordered sets.

Thus for $\alpha = ot(A, \preceq_A)$ and $\beta = ot(B, \preceq_B)$ either $\beta < \alpha$, $\beta = \alpha$,or $\beta > \alpha$.

If $\alpha = \beta$ then clearly $|A| =_c |B|$ since an order isomorphism $A \to B$ is a bijection.

If $\alpha < \beta$ then it means that $(A, \preceq_A)$ is isomorphic to an initial segement of $(B, \preceq_B)$ which implies that $|A| <_c |B|$.

And a similar argument yields that $\alpha > \beta$ implies $|A| >_c |B|$. $\square$

What should be apparent from our development of the ordinals is that, while they represent well-orderings of sets, they too are well ordered under the $<$ relation based upon the comparison of initial segments.

Theorem

Let Φ be a set of ordinal numbers then $(\Phi, <)$ is a well-ordered set.

Proof.

Assume that Φ is non-empty since the empty set is trivially well-ordered.

Let $\phi \in \Phi$ be an ordinal in Φ and if ϕ isn't the first element then consider the segment $s_\Phi(\phi)$ which is well-ordered.

Since $s_\Phi(\phi) \cap \Phi$ is a subset, it is well ordered and so has a first element, call it α.

Therefore α is the first element of Φ so Φ is well ordered. $\square$

Before we consider the enumeration of the ordinals (which we can do, to a point) we come back to a definition we bypassed earlier.

Definition

Let ω^2 be the set $\omega \cdot \omega$ which is built on the set $\omega \times \omega$ with the order $\langle b_1, a_1 \rangle \preceq \langle b_2, a_2 \rangle$ if $b_1 < b_2$, or if $b_1 = b_2$ then $a_1 < a_2$.

This is basically definining the so-called lexicographic or 'dictionary ordering' on $\omega \times \omega$ where

$$\begin{aligned}
&\langle 0,0\rangle < \langle 1,0\rangle < \langle 2,0\rangle < \cdots < \\
&\langle 0,1\rangle < \langle 1,1\rangle < \langle 2,1\rangle < \cdots < \\
&\langle 0,2\rangle < \langle 1,2\rangle < \langle 2,2\rangle < \cdots <
\end{aligned}$$

And inductively we define $\omega^n = \omega^{n-1} \cdot \omega$ although we can simply define it to mean ω^n (n-tuples) of natural numbers with lexicographic ordering.

With this in mind, we can *list* the ordinals starting with $0 = \emptyset$:

$$
\begin{aligned}
&0, 1, \ldots, n, \ldots \\
&\omega, \omega + 1, \ldots, \omega + n, \ldots \\
&\omega \cdot 2, \omega \cdot 2 + 1, \ldots, \omega \cdot 2 + n, \ldots \\
&\vdots \\
&\omega \cdot \omega = \omega^2, \omega^2 + 1, \ldots, \omega^2 + n, \ldots, \omega^2 + \omega, \ldots \\
&\omega^2 \cdot 2, \omega^2 \cdot 2 + 1, \ldots, \omega^2 \cdot 2 + n, \ldots, \omega^2 \cdot 2 + \omega, \ldots \\
&\vdots \\
&\omega^2 \cdot \omega = \omega^3, \omega^3 + 1, \ldots, \omega^3 + n, \ldots \\
&\vdots \text{ and} \\
&\omega^\omega, \ldots
\end{aligned}
$$

We are not going to delve into all the curious properties of ordinal arithmetic, but we can make a few observations.

First, on the previous page we noted the ordinal ω^ω which is viewed as the *limit* of the sequence of exponentiated ordinals $\omega, \omega^2, \dots$ where ω^n is viewed as the set of n-tuples of natural numbers with lexicographic ordering.

This in turn is viewable as the set of all functions $f : \{0, \dots, n-1\} \to \omega$ in that each n-tuple exactly corresponds to the image for $f(\{0, \dots, n-1\})$.

As such, ω^ω is viewed as the set

$$\bigcup_{n\in\omega} \omega^n$$

which is the set of all functions $F : \{0, \ldots, n-1\} \to \omega$ for some $n \in \omega$.

So in particular it is **not** the set of all functions $\omega \to \omega$ since these powers of ω are in a countable hierarchy of ordinal numbers, and indeed ω^ω is still a countable set, being the countable union of countable sets.

In the table of ordinals given above, we organized them in such a way that for a given ordinal α, all the successors $\alpha + n$ for $n \in \omega$ follow it, but that 'above' each of these successors is an ordinal which is **not** a successor.

Definition

An ordinal α is a limit ordinal if it is not of the form $\beta + 1$ for some other ordinal β.

The primary examples in the table are ω, $\omega \cdot 2$,$\omega \cdot 3 \ldots$, ω^2, $\omega^2 \cdot 2, \ldots, \omega^\omega$.

So basically, between any successive limit ordinals α and $\beta = \alpha + \omega$ we have

$$\alpha, \alpha + 1, \ldots, \alpha + \omega$$

where $\alpha + \omega$ has no immediate predecessor.

In the construction of ω^ω as

$$\bigcup_{n\in\omega} \omega^n$$

this union of the chain $\{\omega^n\}$ is what is termed the **supremum** which is actualy what is used to define any limit ordinal since this includes all ordinals in the chain.

Each of these has a successor, but the supremum by construction has no predecessor so is therefore a limit ordinal as per the definition given previously.

What is also rather fascinating is that the ordinals in this hierarchy can be expressed using what is known as **Cantor Normal Form** namely as

$$\omega^{\alpha_1} c_1 + \omega^{\alpha_2} c_2 + \cdots + \omega^{\alpha_k} c_k$$

where k is a natural number, c_i are positive integers and $\alpha_1 > \alpha_2 > \cdots > \alpha_k$ are ordinals.

So we have something analogous to a 'base 10' representation, but rather it's a 'base ω' expression.

Moving farther up the heirarchy, we can contemplate the limit of the exponentiation operation

$$\omega, \omega^{\omega}, \omega^{\omega^{\omega}}, \omega^{\omega^{\omega^{\omega}}}, \ldots$$

to obtain a 'tower' $\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$ which can be thought as the first ordinal ε that satisfies the 'fixed point' equation

$$\varepsilon^{\omega} = \varepsilon$$

namely that under the exponentiation map $\beta \mapsto \omega^{\beta}$ it is a fixed point.

Indeed this one is denoted ε_0, and as it is a countable limit, it is a countable ordinal.

Beyond countable ordinals, we know that every set with a given cardinality can be well-ordered to give rise to an ordinal number.

All the numbers in this list (except for the finite sets $n \in \omega$) are the countable ordinals, and therefore all have cardinality 'aleph-0', that is $\aleph_0$ the cardinality of ω.

As such there is an ordinal number with cardinality $\mathfrak{c}$ namely that of the continuum $\mathbb{R}$.

Moreover, by the well-ordering of any set of ordinals, there must be a 'first' such uncountable ordinal.

We can subdivide the ordinals into various categories or classes according to the cardinality of the underlying set.

We have the finite ordinals of course, and then we have the 'first' infinite ordinal, and we can therefore track the cardinalities in terms of ordinals.

Definition

Let $\aleph_0$ be the cardinality of the set of countable ordinals.
And for each ordinal α, let $\aleph_{\alpha+1}$ be the least ordinal of cardinality *greater* than $\aleph_\alpha$.

But as we observed there must be a first uncountable ordinal which we designate ω_1, and a set A with this ordinal type must be uncountable.

Another tantalizing question is where $\mathfrak{c}$ sit in the sequence of alephs? Our speculation is that $\mathfrak{c} = \aleph_1$ since it's certainly *not* $\aleph_0$ so basically this yet another manifestation of the continuum hypothesis.

Or, framed in terms of power sets, we have that $\mathfrak{c} = 2^{\aleph_0}$ so does this mean that $2^{\aleph_0}$ (the cardinality of the power set of a set of cardinality $\aleph_0$) equals $\aleph_1$?

More generally, the Generalized Continuum Hypothesis, is that statement that

$$\aleph_{k+1} = 2^{\aleph_k}$$

namely that each higher cardinal number is the power set of the previous one.

We know, of course, that a given set and its power set do not have the same cardinality, but this does not necessarily imply the generalized continuum hypothesis above.

To finish the discussion of ordinals, we discuss the following 'paradox' which is reminiscent of the Russell paradox.

The Burali-Forti Antimony, (Cesare Burali-Forti)

If we allow the set Γ of all ordinal numbers, then $(\Gamma, <)$ is well-ordered. Thus Γ has an ordinal number, which we can call γ.

However, we must therefore have $\gamma \in \Gamma$.

So if $\alpha \in \Gamma$ then $\alpha < \gamma$, but for any ordinal α we can form the successor ordinal $\alpha + 1$ which also must lie in Γ and therefore satisfy $\alpha + 1 < \gamma$.

However, this implies that $\gamma + 1 \in \Gamma$ but then we have a contradiction, namely that $\gamma + 1 < \gamma$.

Therefore, no such **set of all ordinals** exists.

However, one does see the use (in the text for example) of the term '*On*' for the class of ordinal numbers, i.e.

$$On := \{\alpha \; : \; \alpha \text{ is an ordinal}\}$$

which is more of a bookkeeping device.

For example, when one wants to take the union of a collection of sets, indexed by ordinal numbers, we can use the *On* to make clear the indices are ordinals.