# MA294 Lecture

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# Generating Functions

Recall, in the ring $F[x]$ (for $F$ a field) the elements are of the form

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

for $a_i \in F$ where addition is performed degree by degree, and multiplication is carried out as follows:

$$(a_0 + a_1 x + \cdots + a_n x^n)(b_0 + b_1 x + \cdots + b_m x^m)$$
$$= (a_0 b_0) + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots + (a_n b_m)x^{n+m}$$

where, in particular, to determine the coefficient of $x^k$ one considers which coefficients 'contribute' in degree $k$, in particular it is:

$$a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0$$

namely the those $a_i b_j$ where $i + j = k$.

As a result, since 1 is the multiplicative identity, one finds that, since $deg(f(x)g(x)) = deg(f(x)) + deg(g(x))$ that

$$U(F[x]) = \{a_0 \mid a_0 \in F^*\}$$

i.e. non-zero constant polynomials

i.e. $F[x]$ doesn't contain inverses for most of its elements!

If we extend our view from polynomials to power series (like in Calculus):

$$a_0 + a_1x + a_2x^2 + \ldots$$

then one has a ring (which contains all polynomials) but with a much richer structure.

## Definition

For $F$ a field, the ring of formal power series is

$$F[[x]] = \{a_0 + a_1 x + a_2 x^2 + \dots \mid a_i \in F\}$$

where addition (like for polynomials) is degree by degree, and multiplication is given by:

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{j=0}^{\infty} b_j x^j\right) = \sum_{k=0}^{\infty} c_k x^k$$

where

$$\begin{aligned} c_0 &= a_0 b_0 \\ c_1 &= a_0 b_1 + a_1 b_0 \\ &\vdots \\ c_k &= \sum_{t=0}^{k} a_t b_{k-t} \end{aligned}$$

The additive identity is $0 + 0x + 0x^2 + \dots$ and multiplicative identity is $1 + 0x + 0x^2 + \dots$ (Exercise!)

Two Important Points:

- A given series $\sum_{i=0}^{\infty} a_i x^i$ is determined by the sequence of coefficients $a_0, a_1, \dots$
- The variable '$x$' is just an indeterminate, we don't consider questions of convergence as one might in calculus, for example, $F[[x]]$ contains series like $1 + x + 2x^2 + 3x^3 + \dots$ which by (say the ratio test) are divergent.

As indicated earlier $F[[x]]$ has a richer structure and contains all the polynomials in $F[x]$ as a subring.

In particular, (and most importantly for the applications we are trying to develop) many polynomials, for example $f(x) = 1 - x$ have inverses when viewed as elements of $F[[x]]$ in sharp contrast with the fact that $U(F[x]) = F^*$.

## Theorem

$\sum_{i=0}^{\infty} a_i x^i \in F[[x]]$ *is invertible if and only if* $a_0 \neq 0$.

**Proof:**

If $\sum_{j=0}^{\infty} b_j x^j$ has the property that $\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{j=0}^{\infty} b_j x^j\right) = 1$ then this implies

$$\begin{aligned} a_0 b_0 &= 1 \\ a_0 b_1 + a_1 b_0 &= 0 \\ a_0 b_2 + a_1 b_1 + a_2 b_0 &= 0 \\ &\vdots \end{aligned}$$

Assuming $a_0 \neq 0$ then $a_0 b_0 = 1$ implies $b_0 = 1/a_0$. So now, let's try and determine $b_1$.

Looking at the second equation $a_0 b_1 + a_1 b_0 = 0$ we use what we've just discovered to rewrite this as

$$a_0 b_1 + a_1(1/a_0) = 0$$

which can be solved to yield $b_1 = (-a_1/a_0^2)$, and we can then solve for $b_2$ by using the third equation:

$$a_0 b_2 + a_1 b_1 + a_2 b_0 = 0$$

which, given that we have already determined $b_0$ and $b_1$ (and we alread know all the $a_i$ since they are given in the first place) we can solve for $b_2$ to get

$$b_2 = a_1^2 a_0^{-2} - a_2 a_0^{-1}$$

and so on...

The point is that $a_0 \neq 0$ implies the existence of $b_0$ and therefore *all* the remaining $b_j$ which therefore yields that

$$\sum_{j=0}^{\infty} b_j x^j$$

is the inverse of $\sum_{i=0}^{\infty} a_i x^i$. $\square$

What we have shown then is that

$$U(F[[x]]) = \{\sum_{i=0}^{\infty} a_i x^i \mid a_0 \neq 0\}$$

which is much 'larger' than $U(F[x])$ in that it contains more than just 'constants'.

How about $\frac{1}{1-x}$? i.e. $(1-x)^{-1}$

Even though $1-x$ is polynomial, it is technically a series, it's just that the coefficients of all terms beyond degree 1 are zero.

And since the constant term of $1-x$ is non-zero, it has an inverse, which (if you think about it) is a series rather than a polynomial, so it will have infinitely many non-zero terms. Let's find it!

$$(b_0 + b_1 x + b_2 x^2 + \dots)(1-x) = 1$$

implies (after distributing the $(1-x)$) that

$$b_0 + (b_1 - b_0)x + (b_2 - b_1)x^2 + \cdots = 1$$

$b_0 + (b_1 - b_0)x + (b_2 - b_1)x^2 + \dots = 1 = 1 + 0x + 0x^2 + \dots$
implies that $b_0 = 1$, and since $b_1 - b_0 = 0$ then $b_1 = 1$, and since $b_2 - b_1 = 0$ then $b_2 = 1$ etc.

Therefore

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

which should be a familiar fact from Calculus! (i.e. geometric series)

And keep in mind, this is formal algebra, we are not concerned about convergence, so we don't 'restrict' $x$ to the interval $(-1, 1)$ which one usually does in that setting.

In general, if $B(x)$ is a polynomial in $F[x]$ then we may regard it as a series in $F[[x]]$ and, if it's invertible, denote its inverse $B(x)^{-1} = \frac{1}{B(x)}$ and if $A(x) \in F[x]$ too, then we have $\frac{A(x)}{B(x)} \in F[[x]]$.

The series for $\frac{A(x)}{B(x)}$ can be computed by long division even though the result is generally a series and not a polynomial.

Consider the example $\frac{1+x}{1+x+x^2}$.

$$
\begin{array}{r|l}
 & 1 \quad - x^2 + x^3 - \quad x^5 + x^6 \quad - x^8 + \dots \\
\hline
1 + x + x^2 & 1 + x \\
 & 1 + x + x^2 \\
\hline
 & \quad - x^2 \\
 & \quad - x^2 - x^3 - x^4 \\
\hline
 & \qquad x^3 + x^4 \\
 & \qquad x^3 + x^4 + x^5 \\
\hline
 & \qquad\quad - x^5 \\
 & \qquad\quad - x^5 - x^6 - x^7 \\
\hline
 & \qquad\qquad x^6 + x^7 \\
 & \qquad\qquad x^6 + x^7 + x^8 \\
\hline
 & \qquad\qquad\quad - x^8 \\
 & \qquad\qquad\qquad \ddots
\end{array}
$$

What this leads to is that $\frac{1+x}{1+x+x^2}$ as a series equals

$$1 - x^2 + x^3 - x^5 + x^6 - x^8 + \dots$$

and similarly, one can show that

$$\begin{aligned}\frac{1}{1+x+x^2} &= 1 - x + x^3 - x^4 + x^6 - x^7 + x^9 - x^{10} + \dots \\ &= (1-x) + (x^3 - x^4) + (x^6 - x^7) + (x^9 - x^{10}) + \dots\end{aligned}$$

Notice a pattern?

# Partial Fractions

$\frac{A(x)}{B(x)}$ is not unfamiliar, recall the technique of partial fractions.

Example: If $B(x) = S(x)T(x)$ where $gcd(S(x), T(x)) = 1$ and $deg(A(x)) < deg(B(x))$ then there exists polynomials $F(x)$, $G(x)$ (where $deg(F) < deg(S)$ and $deg(G) < deg(T)$) such that

$$\frac{A(x)}{B(x)} = \frac{F(x)}{S(x)} + \frac{G(x)}{T(x)}$$

For example, since $2 - 3x + x^2 = (1 - x)(2 - x)$ then

$$\frac{5 - 3x}{2 - 3x + x^2} = \frac{2}{1 - x} + \frac{1}{2 - x}$$

where, indeed, the numerators on the right are lower degree than the denominators.

As is also standard, if one has $\frac{A(x)}{B(x)}$ where $deg(A(x)) > deg(B(x))$ then one divides $A(x)$ by $B(x)$ to get

$$A(x) = Q(x)B(x) + R(x)$$

where $deg(R(x)) < deg(B(X))$ or $R(x) = 0$, which implies

$$\begin{aligned}\frac{A(x)}{B(x)} &= \frac{Q(x)B(x) + R(x)}{B(x)} \\ &= Q(x) + \frac{R(x)}{B(x)}\end{aligned}$$

For example:

$$\begin{aligned}\frac{7-2x-5x^2+2x^3}{2-3x+x^2} &= \frac{(1+2x)(2-3x+x^2)+(5-3x)}{2-3x+x^2}\\ &= (1+2x)+\frac{(5-3x)}{2-3x+x^2}\\ &= (1+2x)+\frac{2}{1-x}+\frac{1}{2-x}\end{aligned}$$

The partial fractions algorithm is based on the following basic fact.

### Theorem

*Let $F$ be a field and $A(x), B(x) \in F[x]$ such that*
*(i)* $deg(A(x)) < deg(B(x))$
*(ii)* $B(x) = S(x)T(x)$ *where* $gcd(S(x), T(x)) = 1$
*(iii)* $B_0 \neq 0$
*Then there are polynomials $f(x)$ and $g(x)$ such that* $deg(f(x)) < deg(S(x))$ *and* $deg(g(x)) < deg(T(x))$ *where*

$$\frac{A(x)}{B(x)} = \frac{f(x)}{S(x)} + \frac{g(x)}{T(x)}$$

*where this equation holds in $F[[x]]$.*

As far as the determination of the numerators, that is a computational problem, fundamentally a linear algebra question.

Example: $\frac{2+3x+x^2}{(1-x)(2+x)(3-x)} = \frac{A}{1-x} + \frac{B}{2+x} + \frac{C}{3-x}$ So how do we find $A, B, C$?

If we multiply the above equation by the denominator on the left, $(1-x)(2+x)(3-x)$ we get the following

$$2 + 3x + x^2 = A(2+x)(3-x) + B(1-x)(3-x) + C(1-x)(2+x)$$

which can be simplified.

$$\begin{aligned}2+3x+x^2 &= A(2+x)(3-x)+B(1-x)(3-x)+C(1-x)(2+x)\\ &= A(6+x-x^2)+B(3-4x+x^2)+C(2-x-x^2)\\ &= (6A+3B+2C)+(A-4B-C)x+(-A+B-C)x^2\end{aligned}$$

So by equating coefficients, this implies

$$\begin{aligned}6A+3B+2C &= 2\\ A-4B-C &= 3\\ -A+B-C &= 1\end{aligned}$$

To solve for $A$,$B$, and $C$ we can employ different strategies, but basically we want to eliminate variables.

So if we add the second and third equations:

$$\begin{aligned} A - 4B - C &= 3 \\ -A + B - C &= 1 \end{aligned}$$

we get $-3B - 2C = 4$.

And if we add $6(-A + B - C = 1)$, namely $-6A + 6B - 6C = 6$ to the *first equation* $6A + 3B + 2C = 2$ we get $9B - 4C = 8$ resulting in the system

$$\begin{aligned} -3B - 2C &= 4 \\ 9B - 4C &= 8 \end{aligned}$$

which can be solved to yield $C = -2$ and $B = 0$, which, in turn, implies $A = 1$.

As such

$$\frac{2+3x+x^2}{(1-x)(2+x)(3-x)} = \frac{A}{1-x} + \frac{B}{2+x} + \frac{C}{3-x}$$
$$= \frac{1}{1-x} - \frac{2}{3-x}$$

and we note that, for examples like the one we just did, where the denominators are all distinct, that one could use a simpler approach to solve for $A$ , $B$, and $C$.

Going back to this stage of the calculation:

$$2 + 3x + x^2 = A(2 + x)(3 - x) + B(1 - x)(3 - x) + C(1 - x)(2 + x)$$

we observe that this is true for all $x \in F$ so, in particular we can let $x = 1$, (which makes $1 - x = 0!$):

$$\begin{array}{c} 2 + 3 \cdot 1 + 1^2 = A(2 + 1)(3 - 1) + B(1 - 1)(3 - 1) + C(1 - 1)(2 + 1) \\ \downarrow \\ 6 = A \cdot 6 \end{array}$$

namely that $A = 1$, and similarly if we let $x = 3$ two of the terms vanish to yield $20 = C \cdot (-10)$ which implies $C = -2$ of course, a nd similarly, letting $x = -2$ yields $B = 0$.

You may recall a variation of the partial fractions algorithm when one has repeated factors.

If $B(x) = P(x)^m$ where $deg(A(x)) < deg(B(x))$ one has

$$\frac{A(x)}{B(x)} = \frac{H_1(x)}{P(x)^1} + \frac{H_2(x)}{P(x)^2} + \cdots + \frac{H_m(x)}{P(x)^m}$$

We consider an example:

$$\frac{x^3+x^2+x+1}{(x-1)(x-2)^3} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3}$$

which is a step up in computational, but not theoretical complexity.
If we do the same 'multiply by the denominator on the left' we get

$$\begin{aligned} x^3+x^2+x+1 = {} & A(x-2)^3 + B(x-1)(x-2)^2 + \\ & C(x-1)(x-2) + D(x-1) \\ & \downarrow \\ & = (A+B)x^3 + (-6A-5B+C)x^2 + \\ & (12A+8B-3C+D)x^2 + (-8A-4B+2C-D) \end{aligned}$$

And if we set $x=2$ we immediately find that $D=15$, and letting $x=1$ we find that $A=-4$ which, since $A+B=1$ yields $B=5$, and, from $-6A-5B+C=1$ that $C=2$.

If one has had linear algebra fairly recently, one *could* solve the previous problem using matrices, namely

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -6 & -5 & 1 & 0 \\ 12 & 8 & -3 & 1 \\ -8 & -4 & 2 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

whose solution

$$\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ 2 \\ 15 \end{pmatrix}$$

is unique (which is guaranteed by the theorem) and agrees with the method used earlier.

# Binomial Theorem for Negative Exponents

Recall the Binomial Theorem:

$$
\begin{aligned}
(y+x)^n &= \sum_{k=0}^{n} \binom{n}{k} y^{n-k} x^k \\
&= \binom{n}{0} y^n + \binom{n}{1} y^{n-1} x + \cdots + \binom{n}{n-1} y x^{n-1} + \binom{n}{n} x^n
\end{aligned}
$$

where, if $y = 1$ we get

$$
(1+x)^n = \binom{n}{0} + \binom{n}{1} x + \cdots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n
$$

Compare this with the geometric series fact:

$$(1+x)^{-1} = \sum_{n=0}^{\infty}(-x)^n = 1 - x + x^2 - x^3 + \dots$$

and that, in $F[[x]]$ this exists, and in general

$$(1+x)^{-m} = (1+x)^{-1} \cdot (1+x)^{-1} \cdots (1+x)^{-1} \ (m\text{-factors})$$

which leads to the following 'extension' or generalization of the binomial theorem.

## Theorem

*The coefficient of* $x^n$ *in the series for* $(1+x)^{-m}$ *is* $(-1)^n\binom{m+n-1}{n}$

For example: In $(1+x)^{-2}$ (i.e. $m=2$) we have

$$(-1)^n\binom{2+n-1}{n} = (-1)^n\binom{n+1}{n}$$

so that

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$$

In general, one can extend the binomial coefficients $\binom{n}{k}$ beyond simply the range $0 \leq k \leq n$.

Observe that normally

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \frac{n(n-1)\cdots 2\cdot 1}{k!(n-k)(n-k-1)\cdots 2\cdot 1}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{k!}$$

So let's define

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

and $\displaystyle\binom{\alpha}{0} = 1$ for all $\alpha$.

This is consistent with the theorem above for the series for $\frac{1}{(1+x)^m}$ since

$$\begin{aligned}\binom{-m}{n} &= \frac{\overbrace{(-m)(-m-1)\cdots(-m-n-1)}^{n\text{-terms}}}{n!} \\ &= (-1)^n\left[\frac{m(m+1)\cdots(m+n-1)}{n!}\right] \\ &= (-1)^n\binom{m+n-1}{n}\end{aligned}$$

We can therefore extend the binomial theorem.

If for $t$ a non-negative integer we define $\binom{t}{n} = 0$ if $n > t$ then

$$(1+x)^t = \binom{t}{0} + \binom{t}{1} x + \binom{t}{2} x^2 + \ldots$$

where now, this is either a polynomial (when $t \geq 0$), or otherwise a series.

Note also that the series:

$$(1-x)^{-m} = 1 + x + \cdots + \binom{m+n-1}{n} x^n + \ldots$$

can be extended a bit to get a series for $(1-ax)^{-m}$ for any $a \in F$ by replacing '$x$' with $ax$ to get:

$$(1-ax)^{-m} = 1 + ax + \cdots + \binom{m+n-1}{n} (ax)^n + \ldots$$

.

And, as mentioned earlier, we can combine these series/polynomials to get series for other rational expressions, for example:

$$
\begin{aligned}
\frac{1+3x}{1-2x+x^2} &= \frac{1+3x}{(1-x)^2} \\
&= (1+3x)(1+2x+3x^2+\cdots+(n+1)x^n+\dots) \\
&= (1+2x+3x^2+\cdots+(n+1)x^n+\dots)+ \\
&\qquad (3x+6x^2+9x^3+\cdots+(3n)x^n+\dots) \\
&= \sum_{n=0}^{\infty}(4n+1)x^n
\end{aligned}
$$

# Square Roots?

One may extend the binomial coefficient $\binom{\alpha}{k}$ to cases where $\alpha$ is a fraction, for example:

$$\binom{\frac{1}{2}}{n} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-n+1)}{n!}$$

which would help us to understand the series representations for something like

$$(1+x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n$$

$$\binom{\frac{1}{2}}{0} = 1$$

$$\binom{\frac{1}{2}}{1} = \frac{\frac{1}{2}}{1!} = \frac{1}{2}$$

$$\binom{\frac{1}{2}}{2} = \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!} = -\frac{1}{8}$$

$$\binom{\frac{1}{2}}{3} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!} = \frac{1}{16}$$

$$\binom{\frac{1}{2}}{4} = -\frac{5}{128}$$

$$\binom{\frac{1}{2}}{5} = \frac{7}{256}$$

As such, we have:

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 + \dots$$

but our natural question is to ask,is this right?
Well, suppose $\left(\sum_{i=0}^{\infty} a_i x^i\right)^2 = 1 + x$, then this would mean

$$\begin{aligned} a_0^2 &= 1 \text{ so } a_0 = 1 \\ a_0a_1 + a_1a_0 &= 1 \text{ so } a_1 = \frac{1}{2} \\ a_0a_2 + a_1a_1 + a_2a_0 &= 0 \text{ so } a_2 = -\frac{1}{8} \\ a_0a_3 + a_1a_2 + a_2a_2 + a_3a_0 &= 0 \text{ so } a_3 = \frac{1}{16} \\ &\vdots \end{aligned}$$

so we have confirmation that the series above, *is* the square-root of $1+x$.

One can also show (as an exercise) that

$$\binom{\frac{1}{2}}{n} = \frac{(-1)^{n-1}}{2^{2n-1}n}\binom{2n-2}{n-1}$$

The point is, $F[[x]]$ contains not just inverses of elements, but radicals of them as well, including finally

$$\begin{aligned}\frac{1}{\sqrt{1+x}} &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4 + \dots \\ &= \sum_{n=0}^{\infty}(-1)^n \binom{\frac{1}{2} - n - 1}{\frac{1}{2} - 1} x^n\end{aligned}$$

which was discovered by Isaac Newton actually.