MA294 Lecture

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Generating Functions

Sequences $\{u_n\}$ for $n = 0, 1, \ldots$, particularly recursively defined ones, can be understood and analyzed by using the terms in the sequence as coefficients of an infinite series

$$U(x) = u_0 + u_1 x + u_2 x^2 + \ldots$$

which is called the (ordinary) generating function for $\{u_n\}$.

We should note that $U(x)$ is not really a function, and again, we are avoiding discussions of convergence as they are irrelevent.

The principle idea is the following:

(i) Take the series $U(x)$ and utilize the 'recurrence relation' (namely the rule which determines how a given u_n depends on the previous terms u_i for $i < n$), to find an equation that $U(x)$ satisfies.

(ii) Use this equation to come up with an 'explicit', that is *non-recursive* formula for the terms $\{u_n\}$. That is, we want to find an explicit function $f(n)$ for $n = 0, 1, \ldots$, such that $u_n = f(n)$.

(iii) The basis for this function is that the equation $U(x)$ satisfies, gives rise to a power series, similar to the geometric (or other) series for example, whose determination will require a fair amount of algebra, including some of the facts about the partial fractions decomposition of a rational function.

The prime example of such a sequence that can be analyze this way is the **Fibonacci Sequence** $\{f_n\}$ defined as follows:

$$f_0 = 0 \;\; f_1 = 1, f_{n+1} = f_n + f_{n-1} \text{ for } n > 1$$

so in particular $f_2 = f_1 + f_0 = 1 + 0 = 1$, $f_3 = f_2 + f_1 = 1 + 1 = 2$, namely $\{0, 1, 1, 2, 3, 5, 8, 13, \dots\}$.

For sequences like the Fibonacci sequence, our goal is to be able to determine the n-the Fibonacci number f_n *without* needing to compute all the f_i for $i < n$. (i.e. what is termed a *closed formula* for f_n)

Without going into too much detail, suffice it to say that the Fibonacci sequence is ubiquitous throughout many areas of mathematics.

As it is a fairly detailed calculation, we shall return to the Fibonacci numbers after examining a number of more modest examples first, to get a feel for the techniques involved.

Consider the sequence $\{u_n\}$ defined by $u_0 = 1$, $u_n = 3u_{n-1}$ where now

$$U(x) = u_0 + u_1x + u_2x^2 + u_3x^3 + u_4x^4 + \dots$$

so the question is, what 'equation' does $U(x)$ satisfy? Observe:

$$\begin{aligned}
U(x) &= u_0 + u_1x + u_2x^2 + u_3x^3 + u_4x^4 + \dots \\
&= u_0 + (3u_0)x + (3u_1)x^2 + (3u_2)x^3 + (3u_3)x^4 + \dots \\
&= u_0 + (3u_0x + 3u_1x^2 + 3u_2x^3 + 3u_3x^4 + \dots) \\
&= u_0 + 3(u_0x + u_1x^2 + u_2x^3 + u_3x^4 + \dots) \\
&= u_0 + 3x(u_0 + u_1x + u_2x^2 + u_3x^3 + \dots) \\
&= 1 + 3xU(x)
\end{aligned}$$

which can be solved to yield $U(x) = \frac{1}{1-3x}$

So now we know that $U(x) = u_0 + u_1 x + \cdots = \frac{1}{1-3x}$ but the key fact which we invoke now is that $\frac{1}{1-3x}$ is expressible as a series, indeed a geometric series.

That is, we know

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

So by replacing 'x' by $3x$ above we get:

$$\begin{aligned}\frac{1}{1-3x} &= 1 + 3x + (3x)^2 + (3x)^3 + \dots \\ &= 1 + 3x + 3^2x^2 + 3^3x^3 + \dots \\ &= \sum_{n=0}^{\infty} 3^n x^n\end{aligned}$$

So we now bridge these two expressions:

$$\begin{aligned} U(x) &= \sum_{n=0}^{\infty} u_n x^n \\ &= \sum_{n=0}^{\infty} 3^n x^n \end{aligned}$$

the conclusion being that $u_n = 3^n$ for *all* n.

This is not a super difficult example in that one *could* observe the pattern that $u_0 = 1$, $u_1 = 3 \cdot 1 = 3$, so $u_2 = 3u_1 = 3 \cdot 3 = 3^2$, etc. but as we'll see, there are more complicated examples where simply computing 'the first few terms' is not quite sufficient to get the explicit formula for u_n for *all* n.

More Generating Function Examples

A simple way to increase the complexity is to add an extra 'constant' term to the recurrence.

Let's take our last example and let $u_0 = 1$ as before, but define $u_n = 3u_{n-1} + 1$ which will be a *different* sequence, but not simply $u_n = 3^n + 1$ (which **isn't** the formula).

Again, we write down a $U(x)$ series and try to determine an equation it satisfies.

$$
\begin{aligned}
U(x) &= u_0 + u_1x + u_2x^2 + u_3x^3 + u_4x^4 + \dots \\
&= u_0 + (3u_0 + 1)x + (3u_1 + 1)x^2 + (3u_2 + 1)x^3 + (3u_3 + 1)x^4 + \dots \\
&= u_0 + (3u_0x + 3u_1x^2 + 3u_2x^3 + 3u_3x^4 + \dots) + (x + x^2 + x^3 + x^4 + . \\
&= u_0 + 3x(u_0 + u_1x + u_2x^2 + u_3x^3 + \dots) + (x + x^2 + x^3 + x^4 + \dots) \\
&= 1 + 3xU(x) + \left[1 + x + x^2 + x^3 + \dots\right] - 1
\end{aligned}
$$

which simplifies to $U(x) = 1 + 3xU(x) + \frac{1}{1-x} - 1$.

Solving for $U(x)$ yields

$$U(x)(1-3x) = 1 + \frac{1}{1-x} - 1$$
$$\downarrow$$
$$U(x) = \frac{1}{(1-3x)(1-x)}$$

So let's determine the coefficients in the series for $U(x)$, based on the series representations of $\frac{1}{1-3x}$ and $\frac{1}{1-x}$.

Using the partial fractions theorem there are constants A, B such that:

$$U(x) = \frac{1}{(1-3x)(1-x)} = \frac{A}{1-3x} + \frac{B}{1-x}$$

which can be solved to yield $A = \frac{3}{2}$, $B = -\frac{1}{2}$ which implies

$$U(x) = \frac{3}{2}\sum_{n=0}^{\infty}(3x)^n - \frac{1}{2}\sum_{n=0}^{\infty}x^n$$

which, since $U(x) = u_0 + u_1 x + \dots$ implies that $u_n = \frac{3}{2}3^n - \frac{1}{2}$ for each n.

So we've determined that $u_n = \frac{3}{2}3^n - \frac{1}{2}$ from this analysis.

It's still good to 'verify' that this is indeed the formula for the n^{th} term.

$u_0 = 1$ implies $u_1 = 3(u_0) + 1 = 4$ and by the forumala

$$u_1 = \frac{3}{2}3^1 - \frac{1}{2} = \frac{9}{2} - \frac{1}{2} = 4$$

etc.

Now let's get back to the Fibonacci sequence, which is a bit more involved, partly because the recrurrence $f_n = f_{n-1} + f_{n-2}$ involves three terms of the sequence as opposed to the two in the example, $u_n = 3u_{n-1}$.

$$\begin{aligned}
F(x) &= f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \dots \\
&= f_0 + f_1 x + (f_0 + f_1)x^2 + (f_1 + f_2)x^3 + (f_2 + f_3)x^4 + \dots \\
&= f_0 + f_1 x + (f_0 x^2 + f_1 x^3 + f_2 x^4 + \dots) + (f_1 x^2 + f_2 x^3 + f_3 x^4 + \dots) \\
&= f_0 + f_1 x + x^2(f_0 + f_1 x + f_2 x^2 + \dots) + x(f_1 x + f_2 x^2 + f_3 x^3 + \dots) \\
&= f_0 + f_1 x + x^2 F(x) + x(F(x) - f_0)
\end{aligned}$$

And if we recall that $f_0 = 0$, $f_1 = 1$ then this boils down to the formula

$$F(x) = x + x^2 F(x) + xF(x)$$

$$\downarrow$$

$$F(x) = \frac{x}{1 - x - x^2}$$

As $F(x) = \frac{x}{1-x-x^2}$ we would like to utilize partial fractions to express this in terms of simple fractions which will yield geometric series of some sort.

The subtle part is factoring the denominator $1 - x - x^2$ as $(1 - ax)(1 - bx)$ and ultimately write

$$F(x) = \frac{A}{1 - ax} + \frac{B}{1 - bx}$$

for some constants A and B.

If $1 - x - x^2 = (1 - ax)(1 - bx) = 1 - (a + b)x + abx^2$ then $a + b = 1$ and $ab = -1$, facts we will use soon.

So
$(1 - ax)(1 - bx) = a(\frac{1}{a} - x)b(\frac{1}{b} - x) = ab(\frac{1}{a} - x)(\frac{1}{b} - x) = ab(x - \frac{1}{a})(x - \frac{1}{b})$
and since $ab = -1$ we get

$$1 - x - x^2 = (1 - ax)(1 - bx) = -(x - \frac{1}{a})(x - \frac{1}{b})$$

Now $1-x-x^2=-x^2-x+1$ has roots given by the quadratic formula, namely

$$\frac{1\pm\sqrt{5}}{-2}=\frac{-1\pm\sqrt{5}}{2}$$

.

So since

$$1-x-x^2=-(x-\frac{1}{a})(x-\frac{1}{b})$$

whose roots are $\frac{1}{a}$ and $\frac{1}{b}$ then we can assume that

$$\frac{1}{a}=\frac{-1+\sqrt{5}}{2} \text{ and } \frac{1}{b}=\frac{-1-\sqrt{5}}{2}$$

which means that

$$a = \frac{2}{-1+\sqrt{5}}$$

$$= \frac{1+\sqrt{5}}{2}$$

$$b = \frac{2}{-1-\sqrt{5}}$$

$$= \frac{1-\sqrt{5}}{2}$$

So..

$$\begin{aligned}\frac{x}{1-x-x^2} &= \frac{x}{(1-ax)(1-bx)} \\ &= \frac{A}{1-ax} + \frac{B}{1-bx} \\ &\downarrow \\ x &= A(1-bx) + B(1-ax)\end{aligned}$$

which can be solved to yield $A = \frac{1}{\sqrt{5}}$, $B = -\frac{1}{\sqrt{5}}$ so that

$$F(x) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} a^n x^n - \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} b^n x^n$$

which implies, ultimately that $f_n = \frac{a^n - b^n}{\sqrt{5}}$

Now, instead of 'a' and 'b' we write $\varphi = \frac{1+\sqrt{5}}{2}$ and $\bar{\varphi} = \frac{1-\sqrt{5}}{2}$ where we refer to φ is the 'golden mean' or 'golden ratio' which we'll talk about more in a moment.

So we've established that $f_n = \frac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}}$ for each $n \geq 0$ and given that $\varphi = \frac{1+\sqrt{5}}{2}$ and $\bar{\varphi} = \frac{1-\sqrt{5}}{2}$ it is perhaps not altogether obvious why

$$\frac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}}$$

is an integer, even though it *must* be as it does equal f_n and each f_n is patently a whole number.

We pause to mention a few interesting facts about φ.

- If one defines $r_n = \frac{f_n}{f_{n-1}}$ for $n \geq 2$ then $\lim\limits_{n\to\infty} r_n = \varphi$.

(Yes, we will need a bit of calculus for this.)

The Monotone Sequence Theorem helps us understand this sequence.

First, I claim that $r_n = \frac{f_n}{f_{n-1}} \leq 2$ for all n.

To see this, suppose the converse, namely that $\frac{f_n}{f_{n-1}} > 2$ which would imply that $f_n > 2f_{n-1}$.

However, since $f_n = f_{n-1} + f_{n-2}$ this would imply that $f_{n-2} > f_{n-1}$ which is impossible since the Fibonacci sequence $\{f_n\}$ is clearly increasing.

Also, we note that $1 \leq r_n$ for each n since $f_n > f_{n-1}$ for $n > 1$.

Therefore $1 \leq r_n \leq 2$.

The sequence r_n is not actually monotonic.

Observe that

$$\begin{array}{lr} r_1 = 1 & r_2 = 2 \\ r_3 = 1.5 & r_4 = 1.\overline{6} \\ r_5 = 1.6 & r_6 = 1.625 \\ r_7 = 1.6154 & r_8 = 1.619 \\ \downarrow & \downarrow \\ \text{increasing} & \text{decreasing} \end{array}$$

and indeed one can show that the 'subsequence' of even terms r_{2k} is decreasing (and, of course, bounded below by 1) and that the odd terms r_{2k+1} are increasing, and bounded above by 2. So we have that $r_{2k} \rightarrow L$ and $r_{2k+1} \rightarrow L'$ for limits L and L'.

Now, the recurrence relation for f_n yields the following:

$$\begin{aligned} r_n &= \frac{f_n}{f_{n-1}} \\ &= \frac{f_{n-1} + f_{n-2}}{f_{n-1}} \\ &= 1 + \frac{1}{r_{n-1}} \end{aligned}$$

So in particular $r_{2k} = 1 + \frac{1}{r_{2k-1}}$ which means

$$\lim_{k\to\infty} r_{2k} = \lim_{k\to\infty} 1 + \frac{1}{r_{2k-1}}$$

$$\downarrow$$

$$L = 1 + \frac{1}{L'}$$

and similarly $r_{2k+1} = 1 + \frac{1}{r_{2k}}$ which means

$$\lim_{k\to\infty} r_{2k+1} = \lim_{k\to\infty} 1 + \frac{1}{r_{2k}}$$

$$\downarrow$$

$$L' = 1 + \frac{1}{L}$$

and so $LL' = L' + 1$ and $LL' = L + 1$ which means $L = L'$ so the even and odd terms both converge to the same number.

So if $L = \lim_{n\to\infty} r_n$ then we find that

$$L = \lim_{n\to\infty} r_n = \lim_{n\to\infty}(1 + \frac{1}{r_{n-1}}) = 1 + \lim_{n\to\infty} \frac{1}{r_{n-1}} = 1 + \frac{1}{L} \text{ !!}$$

which implies $L = 1 + \frac{1}{L}$ namely $L^2 - L - 1 = 0$ so that L is either

$$\frac{1+\sqrt{5}}{2} \text{ or } \frac{1-\sqrt{5}}{2}$$

but since $\frac{1-\sqrt{5}}{2} < 0$ and all $r_n > 0$ then $L = \frac{1+\sqrt{5}}{2} = \Phi$.

Another neat identity that one can also show (yes using Calculus):

- $\varphi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}$

namely the limit of the sequence $\{s_n\}$ where $s_1 = 1$, $s_n = \sqrt{1 + s_{n-1}}$.

We can show that each $s_n < 2$. How?

Well, $s_1 = 1 < 2$ so assume $s_{n-1} < 2$ then

$$s_n = \sqrt{1 + s_{n-1}} < \sqrt{1+2} = \sqrt{3} < 2$$

which is proof by *induction* that all $s_n < 2$. Also it is very clear that $\{s_n\}$ is increasing so by MST the sequence has a limit L which can be interpreted as that 'infinitely nested' square root above.

To show that $L = \varphi$, consider this:

$$\begin{aligned} L &= \sqrt{1+\sqrt{1+\sqrt{1+\dots}}} \\ &\downarrow \\ L^2 &= 1+\sqrt{1+\sqrt{1+\dots}} \\ &= 1+L \end{aligned}$$

and so $L^2 - L - 1 = 0$ which again implies that $L = \frac{1+\sqrt{5}}{2} = \Phi$.

Neat Fact: If we change the value of s_0 to *any* value bigger than -1 the limit is still Φ!

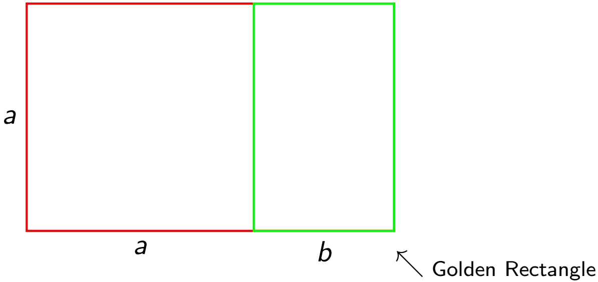
```
perl -e ’$s=1;for($i=1;$i<=5;$i++){$s=sqrt(1+$s);print ‘‘$s\n"}’
1.4142135623731
1.55377397403004
1.59805318247862
1.61184775412525
1.61612120650812

perl -e ’$s=2;for($i=1;$i<=5;$i++){$s=sqrt(1+$s);print "$s\n’’}’
1.73205080756888
1.65289165028107
1.62876998077723
1.62134819849939
1.61905781196948

perl -e ’$s=0;for($i=1;$i<=5;$i++){$s=sqrt(1+$s);print "$s\n’’}’
1
1.4142135623731
1.55377397403004
1.59805318247862
1.61184775412525

perl -e ’$s=-0.5;for($i=1;$i<=5;$i++){$s=sqrt(1+$s);print "$s\n’’}’
0.707106781186548
1.30656296487638
1.51873729291026
1.58705302145526
1.60843185166648
```

- $\varphi = 1 + \frac{1}{\varphi}$ and $\bar{\varphi} = -\frac{1}{\varphi}$
- If $\frac{a}{a+b} = \frac{b}{a}$ then we have that $\frac{a+b}{a} = \varphi \approx 1.618$.



Another useful tool for working with infinite series, especially those arising from fractions $\frac{A(x)}{B(x)}$ is this:

Theorem

If $f(x) \in F[[x]]$ where $f(x)$ is a polynomial which is invertible, then if

$$\frac{1}{f(x)} = \sum_{n=0}^{\infty} c_n x^n$$

then

$$\begin{aligned}\frac{d}{dx}\frac{1}{f(x)} &= \frac{d}{dx}\sum_{n=0}^{\infty} c_n x^n \\ &= \frac{d}{dx}(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots) \\ &= c_1 + 2c_2 x + 3c_2 x^2 + \dots \\ &= \sum_{n=1}^{\infty} n c_n x^{n-1}\end{aligned}$$

That is, we can differentiate series in $F[[x]]$ and the 'function' represented by the derivative is given by the series obtained from differentiating the terms one-by-one.

This should be somewhat familiar if you recall the development of series in Calculus 2.

However, here we are still not assuming anything 'analytic' in the usage of this theorem, since it is actually a statement about algebra since the functions being represented by series are 'algebraic', i.e. polynomials or rational functions.

Example:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$$

$$\downarrow$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + \dots$$

$$\downarrow$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n$$

To see that $1 + 2x + 3x^2 + \dots$ really *does* represent $\frac{1}{(1-x)^2}$ simply compute $(1-x)^2(1 + 2x + 3x^2 + \dots)$

$$
\begin{aligned}
(1-x)^2\left(\sum_{n=0}^{\infty}(n+1)x^n\right) &= (1-2x+x^2)\left(\sum_{n=0}^{\infty}(n+1)x^n\right) \\
&= \sum_{n=0}^{\infty}(n+1)x^n - 2x\left(\sum_{n=0}^{\infty}(n+1)x^n\right) + x^2\left(\sum_{n=0}^{\infty}(n+1)x^n\right) \\
&= \sum_{n=0}^{\infty}(n+1)x^n - \left(\sum_{n=0}^{\infty}(2n+2)x^{n+1}\right) + \left(\sum_{n=0}^{\infty}(n+1)x^{n+2}\right) \\
&= (1 + 2x + 3x^2 + 4x^3 + \dots) \\
&- (2x + 4x^2 + 6x^3 + \dots) \\
&+ (1x^2 + 2x^3 + 3x^4 + \dots) \\
&= 1
\end{aligned}
$$

Here is a nice example with a sequence $\{u_n\}$ defined by the recurrence: $u_0 = 0,\ u_{n+1} = u_n + (n+1)$.

$$
\begin{aligned}
U(x) &= u_0 + u_1 x + u_2 x^2 + u_3 x^3 + \dots \\
&= u_0 + (u_0 + 1)x + (u_1 + 2)x^2 + (u_2 + 3)x^3 + \dots \\
&= u_0 + x(u_0 + u_1 x + \dots) + (x + 2x^2 + 3x^3 + \dots) \\
&= u_0 + x(u_0 + u_1 x + \dots) + x(1 + 2x + 3x^2 + \dots) \\
&= u_0 + xU(x) + x\left(\frac{1}{(1-x)^2}\right) \\
&= xU(x) + x\left(\frac{1}{(1-x)^2}\right)
\end{aligned}
$$

So we have that $U(x) - xU(x) = \dfrac{x}{(1-x)^2}$ which means $U(x) = \dfrac{x}{(1-x)^3}$.

The question is now, what is the series which represents $U(x)$?

Bear in mind that

$$\frac{d}{dx}\frac{1}{(1-x)^2} = \frac{2}{(1-x)^3}$$

so

$$\begin{aligned}
\frac{2}{(1-x)^3} &= \frac{d}{dx}\sum_{n=0}^{\infty}(n+1)x^n \\
&= \sum_{n=1}^{\infty} n(n+1)x^{n-1} \\
&= \sum_{n=0}^{\infty}(n+1)(n+2)x^n \\
&\downarrow \text{ multiply by } \frac{x}{2} \\
\frac{x}{(1-x)^3} &= \sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^{n+1} \\
&= 0 + \frac{1(2)}{2}x + \frac{2(3)}{2}x^2 + \frac{3(4)}{2}x^3 + \ldots \\
&= u_0 + u_1x + u_2x^2 + u_3x^3 + \ldots
\end{aligned}$$

So, in the final analysis, the recurrence $u_0 = 0$ and $u_{n+1} = u_n + (n+1)$, which implies

- $u_0 = 0$
- $u_1 = 0 + 1 = 1$
- $u_2 = 1 + 2$
- $u_3 = 1 + 2 + 3$
- etc.

which results in the familiar formula for the 'Triangular Numbers'

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

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