MA294 Lecture

Timothy Kohl

Boston University

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Homogeneous Linear Recurrences

An HLR is a sequence $\{u_n\}$ where $u_0 = c_0$, $u_1 = c_1, \ldots, u_{k-1} = c_{k-1}$ for constants $c_0, \ldots, c_{k-1}$ and

$$u_{n+k} + (a_1 u_{n+k-1} + \cdots + a_k u_n) = 0$$

for $n \geq 0$, namely that u_{n+k} is expressed as a linear combination of the k previous terms in the sequence.

For example, for the Fibonacci sequence $f_0 = 1$, $f_1 = 1$, (whence $k = 2$) and $f_{n+2} = f_{n+1} + f_n$, namely

$$f_{n+2} - 1f_{n+1} - 1f_n = 0$$

namely $a_1 = -1$ and $a_2 = -1$.

The usage of the term 'homogeneous' is due to the recurrence relation being expressed as $u_{n+k} + (a_1 u_{n+k-1} + \cdots + a_k u_n) = 0$, i.e. the right hand side being 0.

As such, the example earlier $u_0 = 1$, $u_n = 3u_{n-1} + 1$ is *not* homogeneous since $u_n - 3u_{n-1} = 1$.

You may say, what if we subtract 1 to get $u_n - 3u_{n-1} - 1 = 0$?

But the resulting equation on the left is not a linear combination of the sequence terms as it was for Fibonacci.

Theorem

For a sequence $\{u_n\}$ defined by an HLR as above, the generating function is

$$U(x) = \frac{R(x)}{1 + a_1 x + \cdots + a_k x^k}$$

where $R(x)$ is a polynomial where $deg(R(x)) < k$.

Proof.

$$(1 + a_1 x + \cdots + a_k x^k) U(x) =$$
$$(1 + a_1 x + \cdots + a_k x^k)(u_0 + u_1 x + \cdots + u_n x^n + \dots)$$

where the coefficient of x^{n+k} (for any $n \geq 0$) is
$u_{n+k} + a_1 u_{n+k-1} + \cdots + a_k u_n$ which is **zero** since
$u_{n+k} + (a_1 u_{n+k-1} + \cdots + a_k u_n) = 0$ by assumption.
As such, the right hand side is a polynomial of degree $< k$. $\square$

From the recurrence relation itself $u_{n+k} + a_1 u_{n+k-1} + \cdots + a_k u_n = 0$ we define the auxiliary equation

$$t^k + a_1 t^{k-1} + \cdots + a_k = 0$$

and if this can be factored in F (say $F = \mathbb{C}$) as

$$(t - \alpha_1)^{m_1}(t - \alpha_2)^{m_2} \cdots (t - \alpha_s)^{m_s}$$

then one gets

$$U(x) = \frac{R(x)}{(1 - \alpha_1 x)^{m_1}(1 - \alpha_2 x)^{m_2} \cdots (1 - \alpha_s x)^{m_s}}$$

where each factor in the denominator can treated using the binomial theorem (i.e. as a geometric series).

The end result is that

$$u_n = P_1(n)\alpha_1^n + \cdots + P_s(n)\alpha_s^n$$

where $P_i(n) = A_0 + A_1 n + \cdots + A_{m_i-1} n^{m_i-1}$.

Let's recapitulate two earlier examples.

$u_0 = 1$, $u_{n+1} = 3u_n$, so that $u_{n+1} - 3u_n = 0$, i.e. $a_1 = -3$

So the auxiliary equation is $t - 3 = 0$ which has one root $\alpha = 3$ of course.

Thus $P_1(n) = A$ (which is degree 0 since $deg(t-3) = 1$).

So $u_n = A3^n$ for each n and since $u_0 = 1$ $(n = 0)$ this implies $A = 1$.

That is $u_n = 3^n$ for all n, which agrees with our analysis earlier using geometric series.

Now let's revisit the Fibonacci sequence, where this technique demonstrates its utility.

$u_{n+2} + (-1)u_{n+1} + (-1)u_n = 0$ so that $a_1 = -1$ and $a_2 = -1$

So the auxiliary equation is $t^2 - t - 1 = 0$ where now

$$t^2 - t - 1 = (t - \phi)(t - \bar{\phi})$$

where $\phi, \bar{\phi} = \frac{1 \pm \sqrt{5}}{2}$ and $m_1 = 1,\ m_2 = 1$.

Thus $P_1(n) = A$ and $P_2(n) = B$ since $m_1 - 1 = 0$ and $m_2 - 1 = 0$.

So $u_n = A\phi^n + B\bar{\phi}^n$ for each n and so to find A, B we proceed as follows:

$$
\begin{aligned}
0 &= A\phi^0 + B\bar{\phi}^0 = A + B \rightarrow \ A = -B \\
1 &= A\phi^1 + B\bar{\phi}^1 \\
&= A\phi + B\bar{\phi} \\
&= A(\frac{1+\sqrt{5}}{2}) + B(\frac{1-\sqrt{5}}{2}) \\
&= A(\frac{1+\sqrt{5}}{2}) - A(\frac{1-\sqrt{5}}{2}) \\
&= A(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}) \\
&= A(\frac{2\sqrt{5}}{2}) \\
&= A(\sqrt{5})
\end{aligned}
$$

and thus $A = \frac{1}{\sqrt{5}}$ and $B = -\frac{1}{\sqrt{5}}$.

So we conclude (as we already demonstrated previously) that

$$f_n = \frac{1}{\sqrt{5}}\phi^n - \frac{1}{\sqrt{5}}\bar{\phi}^n$$

Inhomogeneous Recursion Relations

For example: $r_0 = 0,\ r_1 = 0,\ r_{n+2} = 3r_{n+1} + 4r_n + (n+2)$ for $n \geq 2$.

Here we define:

$$
\begin{aligned}
R(x) &= r_0 + r_1 x + r_2 x^2 + r_3 x^3 + \dots \\
&= r_0 + r_1 x + (3r_1 + 4r_0 + 3)x^2 + (3r_2 + 4r_1 + 4)x^3 + \dots \\
&= r_0 + r_1 x + (3r_1 x^2 + 3r_2 x^3 + \dots) + (4r_0 x^2 + 4r_1 x^3 + \dots) + (3x^2 + 4x^3 + 5x^4 \\
&\downarrow\ 3xR(x) = 3r_0 x + 3r_1 x^2 + \dots \\
&\downarrow\ 4x^2 R(x) = 4r_0 x^2 + 4r_1 x^3 + \dots \\
&= 3xR(x) + 4x^2 R(x) + (3x^2 + 4x^3 + \dots) \\
&= 3xR(x) + 4x^2 R(x) + \left[\frac{1}{(1-x)^2} - (1+2x)\right]
\end{aligned}
$$

So we have that

$$R(x)(1-3x-4x^2) = \frac{1}{(1-x)^2} - (1+2x) = \frac{3x^2-2x^3}{(1-x)^2}$$

and so we solve for $R(x)$ to get

$$\begin{aligned} R(x) &= \frac{x^2(3-2x)}{(1-4x)(1+x)(1-x)^2} \\ &= \frac{A}{1-4x} + \frac{B}{1+x} + \frac{C}{1-x} + \frac{D}{(1-x)^2} \end{aligned}$$

$$\frac{A}{1-4x}+\frac{B}{1+x}+\frac{C}{1-x}+\frac{D}{(1-x)^2}$$

$$=\frac{2}{9}\left(\frac{1}{1-4x}\right)+\frac{1}{4}\left(\frac{1}{1+x}\right)-\frac{11}{36}\left(\frac{1}{1-3x}\right)-\frac{1}{6}\left(\frac{1}{(1-x)^2}\right)$$

To complete the analysis we use the following series representations:

$$\frac{1}{1-4x} = \sum_{n=0}^{\infty}(4x)^n = \sum_{n=0}^{\infty} 4^n x^n$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n$$

This implies that

$$R(x) = \sum_{n=0}^{\infty} \underbrace{\left[\left(\frac{2}{9}\right)4^n + \left(\frac{1}{4}\right)(-1)^n - \left(\frac{11}{36}\right)1^n - \left(\frac{1}{6}(n+1)\right)\right]}_{r_n} x^n$$

which is not something one would 'guess' or be easily able to determine from looking at the first few terms.

Here is a slightly different example, but still uses many of the same techniques.

Recall the triangular numbers formula

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

for each integer $n \geq 1$.

This is usually proven by induction, but as we saw, this can be done with generating functions.

What if we want the analogous formula for squares, namely:

$$1^2 + 2^2 + \cdots + n^2 = f(n)$$

for some function f.

To handle this case, we will use utilize the well known fact that $\frac{d}{dx}x^n = nx^{n-1}$, which we used earlier to derive the fact that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\downarrow$$

$$\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$$

$$\downarrow$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$$

but we utilize the fact that the above method is true for finite versions of the sum on the right.

You may have seen this fact given in Calculus,

$$1 + x + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}$$

provided $x \neq 1$. We can manipulate both sides of this equation.

$$1+x+\cdots+x^{n-1}=\frac{x^n-1}{x-1}$$

$$\downarrow$$

$$\frac{d}{dx}\left(1+x+\cdots+x^{n-1}\right)=\frac{d}{dx}\left(\frac{x^n-1}{x-1}\right)$$

$$\downarrow$$

$$1+2x+3x^2+\cdots+(n-1)x^{n-2}=\frac{-x^{n-1}n+1+(n-1)\,x^n}{(x-1)^2}$$

$$\downarrow \text{ multiply by } x$$

$$x+2x^2+3x^3+\cdots+(n-1)x^{n-1}=\frac{-x^n n+x+(n-1)\,x^{n+1}}{(x-1)^2}$$

$$x + 2x^2 + 3x^3 + \cdots + (n-1)x^{n-1} = \frac{-x^n n + x + (n-1)\, x^{n+1}}{(x-1)^2}$$

$$\downarrow$$

$$\frac{d}{dx}\left(x + 2x^2 + 3x^3 + \cdots + (n-1)x^{n-1}\right) = \frac{d}{dx}\left(\frac{-x^n n + x + (n-1)\, x^{n+1}}{(x-1)^2}\right)$$

$$\downarrow$$

$$1 + 4x + 9x^2 + \cdots + (n-1)^2 x^{n-2} = \frac{((n-2)\, x - n)\,(x-1)\, n x^{n-1} + (x^n - 1)\,(x+1)}{(1-x)^3}$$

At this stage it looks very messy, but if we can 'let $x = 1$' on both sides, in particular the left we get

$$1 + 4 + 9 + \cdots + (n-1)^2 = 1^1 + 2^2 + 3^2 + \cdots + (n-1)^2$$

but letting $x = 1$ on the right yields the indeterminate form $\frac{0}{0}$.

So, even though we insisted that this theory does not involve calculus, we can make an exception in this case since it will very readily give us our answer.

Basically

$$1+4x+9x^2+\cdots+(n-1)^2x^{n-1} = \frac{\left(\left(n-2\right)x-n\right)\left(x-1\right)nx^{n-1}+\left(x^n-1\right)\left(x+1\right)}{(1-x)^3}$$

$$\downarrow$$

$$\lim_{x\to 1} 1+4x+9x^2+\cdots+(n-1)^2x^{n-1} = \lim_{x\to 1}\frac{\left(\left(n-2\right)x-n\right)\left(x-1\right)nx^{n-1}+\left(x^n-1\right)\left(x+1\right)}{(1-x)^3}$$

$$\downarrow \text{ L'Hopitals Rule}$$

$$1^2+2^2+\cdots+(n-1)^2 = \frac{(n-1)n(2n-1)}{6}$$

which, implies

$$1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

for each $n\geq 1$.

Exponential Functions and Counting

The binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

yields a number of related interesting formulæ, such as what happens if one chooses $a = 1$ and $b = 1$ for then

$$(1+1)^n = 2^n = \sum_{k=0}^{n} \binom{n}{k}$$

which, of course, has a standard combinatorial interpretation as the size of the power set of any set X of cardinality n, where each binomial coefficient $\binom{n}{k}$ corresponds to the number of subsets of X with k elements.

Consider now the Taylor series for e^x centered at $x = 0$,

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

which, if multiplied by itself, yields

$$\begin{aligned}
e^{2x} &= (\sum_{m=0}^{\infty} \frac{1}{m!} x^m)(\sum_{n=0}^{\infty} \frac{1}{n!} x^n) \\
&= \sum_{q=0}^{\infty} \left(\sum_{i=0}^{q} \frac{1}{i!} \frac{1}{(q-i)!} \right) x^q \\
&= \sum_{q=0}^{\infty} \left(\sum_{i=0}^{q} \frac{1}{q!} \binom{q}{i} \right) x^q \\
&= \sum_{q=0}^{\infty} \left(\frac{(\sum_{i=0}^{q} \binom{q}{i})}{q!} \right) x^q
\end{aligned}$$

Looking at the last expression:

$$\sum_{q=0}^{\infty}\left(\frac{\left(\sum_{i=0}^{q}\binom{q}{i}\right)}{q!}\right)x^q$$

one should look at the coefficient $\left(\frac{\left(\sum_{i=0}^{q}\binom{q}{i}\right)}{q!}\right)$

The numerator $\sum_{i=0}^{q}\binom{q}{i})$ can be seen to be 2^q by simply replace x with $2x$ in the usual Taylor series for e^x, or by computing it via Taylor's Theorem.

If we compute this Taylor series in the usual fashion, utilizing that fact that the q-derivative of e^{2x} is $2^q e^{2x}$, we get

$$e^{2x} = \sum_{q=0}^{\infty} \frac{2^q}{q!} x^q$$

and we conclude, by comparing the two series, that

$$\sum_{i=0}^{q} \binom{q}{i} = \sum_{i=0}^{q} \frac{q!}{i!(q-i)!} = 2^q$$

If we repeat this with the product $e^{3x} = e^x \cdot e^x \cdot e^x$ then we have

$$
\begin{aligned}
e^{3x} &= (\sum_{m=0}^{\infty} \frac{1}{m!} x^m)(\sum_{n=0}^{\infty} \frac{1}{n!} x^n)(\sum_{p=0}^{\infty} \frac{1}{p!} x^p) \\
&= \sum_{q=0}^{\infty} \left(\sum_{i,j,k | i+j+k=q} \frac{1}{i!} \frac{1}{j!} \frac{1}{k!} \right) x^q
\end{aligned}
$$

where we observe (or define if you haven't seen this before) the multinomial coefficient:

$$\binom{q}{i, j, k} = \frac{q!}{i!j!k!}$$

where $i + j + k = q$, which we pause to observe is the coefficient of $x^i y^j z^k$ in $(x + y + z)^q$, which is the so-called multinomial theorem.

Now back to the computation:

$$
\begin{aligned}
e^{3x} &= \sum_{q=0}^{\infty}\left(\sum_{i,j,k|i+j+k=q}\frac{1}{i!}\frac{1}{j!}\frac{1}{k!}\right)x^q \\
&= \sum_{q=0}^{\infty}\left(\sum_{i,j,k|i+j+k=q}\frac{1}{q!}\binom{q}{i,j,k}\right)x^q \\
&= \sum_{q=0}^{\infty}\left(\frac{\sum_{i,j,k|i+j+k=q}\binom{q}{i,j,k}}{q!}\right)x^q
\end{aligned}
$$

And, in comparison, the Taylor series of e^{3x} derived in the ordinary way yields

$$e^{3x} = \sum_{q=0}^{\infty} \frac{3^q}{q!} x^q$$

so that we have another identity

$$\sum_{i,j,k | i+j+k=q} \binom{q}{i,j,k} = 3^q$$

and similarly for higher order multinomial expressions.