MA294 Lecture

Timothy Kohl

Boston University

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One interpretation of the formula relating the series for e^{2x} and the binomial coefficients is to view the following sequence

$$b_n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

as having e^{2x} as its 'generating function' although it doesn't show up as the coefficient of x^n but rather as the coefficient of $\frac{x^n}{n!}$, i.e.

$$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n = \sum_{n=0}^{\infty} 2^n \frac{x^n}{n!}$$

Indeed, if we go back to the multiplication of $e^x \cdot e^x = e^{2x}$ the coefficient of x^n is the result of the contributions of the coefficients of the powers $x^i \cdot x^j$ where $i + j = n$ and these are precisely

$$\frac{x^i}{i!}\frac{x^j}{j!} = \frac{x^{i+j}}{i!j!}$$

and for $i + j = n$ we can express this as $\frac{n!}{i!j!}\frac{x^n}{n!}$ which is precisely $\binom{n}{i}\frac{x^n}{n!}$.

The tie-in with combinatorics/set theory is that the identity

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

is interpreted as the right hand side being the size of the power set of a set with n elements, and the left hand side is the number of ways each subset of size k from $k = 0, \ldots n$.

Exponential Generating Functions

This idea of looking at sequence terms as coefficients of $\frac{x^n}{n!}$ is not as weird as it first seems.

Definition

Given a sequence $\{a_n\}$, the series

$$A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

exponential generating function of the sequence $\{a_n\}$.

For example, the sequence $\{1,1,1,\dots\}$ has the series

$$\sum_{n=0}^{\infty} 1\frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

as its generating function which, of course, is just the series for e^x.

And similarly $\{(-1)^n\}_{n=0}^{\infty}$ has the e.g.f.

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

which is recognizably the series for e^{-x}.

One of the nice properties of exponential generating functions is how they behave under differentiation, in particular because $\frac{d}{dx}\frac{x^n}{n!} = \frac{x^{n-1}}{(n-1)!}$.

Lemma

If $A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = a_0 + a_1\frac{x}{1!} + a_2\frac{x^2}{2!} + \ldots$ is the e.g.f. for $\{a_n\}_{n=0}^{\infty}$ then $A'(x) = \sum_{n=1}^{\infty} a_n \frac{x^{n-1}}{(n-1)!}$ is the e.g.f. of $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, \ldots\}$.

The proof is pretty obvious.

The outcome of this is a way to pass from the sequence $\{a_0, a_1, \ldots\}$ to the 'shifted' sequence $\{a_1, a_2, \ldots\}$.

And it is easy to see that the second derivative $A''(x) = \sum_{n=2}^{\infty} a_n \frac{x^{n-2}}{(n-2)!}$ is the e.g.f. of the sequence $\{a_n\}_{n=2}^{\infty} = \{a_2, a_3, \ldots\}$.

This actually leads to yet another way to re-capitulate the Fibonacci sequence.

For $\{f_n\}$ where $f_0 = 0, f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ if $F(x)$ is the e.g.f. for $\{f_n\}_{n=0}^{\infty}$ then we have

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} = f_0 + f_1\frac{x}{1!} + f_2\frac{x^2}{2!} + f_3\frac{x^3}{3!} + f_4\frac{x^4}{4!} + \dots$$

and as with the ordinary generating function analysis, we utilize the recurrence relation to derive information about $F(x)$.

However, instead of an algebraic equation involving $F(x)$ we instead will have a *differential* equation.

$$
\begin{aligned}
F(x) &= f_0 + f_1\frac{x}{1!} + f_2\frac{x^2}{2!} + f_3\frac{x^3}{3!} + f_4\frac{x^4}{4!} + \dots \\
&\downarrow \\
F'(x) &= f_1 + f_2\frac{x}{1!} + f_3\frac{x^2}{2!} + f_4\frac{x^3}{3!} + \dots \\
&\downarrow \\
F''(x) &= f_2 + f_3\frac{x}{1!} + f_4\frac{x^2}{2!} + \dots \\
&= (f_1 + f_0) + (f_2 + f_1)\frac{x}{1!} + (f_3 + f_2)\frac{x^2}{2!} + \dots \\
&= f_1 + f_2\frac{x}{1!} + f_3\frac{x^2}{2!} + \dots \\
&+ f_0 + f_1\frac{x}{1!} + f_2\frac{x^2}{2!} + \dots \\
&= F'(x) + F(x)
\end{aligned}
$$

So we have the differential equation $F''(x) = F'(x) + F(x)$ which is a 2^{nd} order linear differential equation.

By standard theory, it is known that $F(x)$ is a linear combination of one or more exponential functions e^{rx} which, in this case implies that each such e^{rx} must satisfy this equation.

i.e.

$$\begin{gathered} F''(x) = F'(x) + F(x) \\ \downarrow \\ r^2 e^{rx} = re^{rx} + e^{rx} \end{gathered}$$

That is $(r^2 - r - 1)e^{rx} = 0$.

Since $(r^2 - r - 1)e^{rx} = 0$ then since e^{rx} is *never* zero, then $r^2 - r - 1 = 0$ (look familiar?) which implies that $r = \phi, \bar{\phi}$, namely the golden mean and its conjugate radical.

Moreover, this implies that $F(x) = ae^{\phi x} + be^{\bar{\phi}x}$, namely that $F(x)$ is a linear combination of exactly two such basic exponentials e^{rx}. Why?

Well, as seen above if any e^{rx} satisfies the differential equation then r is root of $x^2 - x - 1 = 0$, but this equation can only have two roots.

As to the solution of $F(x) = ae^{\phi x} + be^{\bar{\phi}x}$ we may, for example, let $x = 0$ which yields

$$F(0) = ae^0 + be^0 = a + b$$

where, since $F(x) = \sum_{n=0} f_n \frac{x^n}{n!} = f_0 + f_1 \frac{x}{1!} + \dots$, implies that $F(0) = f_0 = 0$ so $a + b = 0$.

If we take derivatives we get $F'(x) = f_1 + f_2 \frac{x}{1!} + \dots$ but also $F'(x) = a\phi e^{\phi x} + b\bar{\phi}e^{\bar{\phi}x}$ which implies $F'(0) = f_1 = 1 = a\phi + b\bar{\phi}$

This may be familiar from before, but let's finish the calculation

$$a + b = 0$$
$$a\frac{1+\sqrt{5}}{2} + b\frac{1-\sqrt{5}}{2} = 1$$

which implies that $a = \frac{1}{\sqrt{5}}$ and $b = -\frac{1}{\sqrt{5}}$.

So we have, using the 'a' and 'b' notation for the moment:

$$\begin{aligned} F(x) &= f_0 + f_1\frac{x}{1!} + f_2\frac{x^2}{2!} + f_3\frac{x^3}{3!} + \dots \\ &= ae^{\phi x} + be^{\bar{\phi}x} \\ &= a(1 + \frac{\phi x}{1!} + \frac{\phi^2 x^2}{2!} + \frac{\phi^3 x^3}{3!} + \dots \\ &+ b(1 + \frac{\bar{\phi} x}{1!} + \frac{\bar{\phi}^2 x^2}{2!} + \frac{\bar{\phi}^3 x^3}{3!} + \dots \end{aligned}$$

which implies, immediately that $f_n = a\phi^n + b\bar{\phi}^n = \frac{1}{\sqrt{5}}\phi^n - \frac{1}{\sqrt{5}}\bar{\phi}^n$

A Small Excursion - The Weyl Algebra

The usage of the operations of differentiation $\frac{d}{dx}$ and 'multiplication by x' played a role in quite a number of calculations we did in the study of generating functions.

If for the ring $F[[x]]$ (or even the ring $F[x]$) we let D denote the operation of differentiation, that is

$$D(f(x)) = \frac{d}{dx}f(x)$$

and if we let X denote the operation of multiplication by x, that is $X(f(x)) = x \cdot f(x)$ then

$$\begin{aligned} D &: F[[x]] \to F[[x]] \\ X &: F[[x]] \to F[[x]] \end{aligned}$$

are actually linear transformations, if we view $F[[x]]$ as a vector space over F.

This is also true if we restrict D, X to $F[x]$.

The linearity of D is a fact one learns in calculus, namely that $D(f(x)+g(x)) = D(f(x)) + D(g(x))$ and $D(c \cdot f(x)) = cD(f(x))$ for $c \in F$ a constant.

i.e. 'The derivative of a sum is ...'

But X is also linear in that

$$X(f(x)+g(x)) = xf(x) + xg(x) = x(f(x)+g(x))$$

and that $X(cf(x)) = c \cdot x \cdot f(x) = x \cdot (c \cdot f(x))$ for any series (or polynomials) $f(x), g(x)$.

What is of interest to us is how these operations interact with each other.

Observe that

$$XD(f(x)) = xf'(x)$$

whereas, by virtue of the product rule (which can be viewed purely algebraically),

$$DX(f(x)) = D(xf(x)) = 1 \cdot f(x) + xf(x) = f(x) + X(f(x)) = I(f(x)) + X(f(x))$$

where $I : F[[x]] \to F[[x]]$ is the identity mapping which is also, clearly, a linear transformation.

So what we have discovered, is that

$$DX = I + XD$$

or, more perspicaciously,

$$DX - XD = I$$

so, these operations **do not** commute.

This leads to the following:

Definition

For X, D defined above, the Weyl algebra W is the ring consisting of all sums and products of powers of X and D, where $D^i(f(x)) = \frac{d^i f}{dx^i}$ and $X^j(f(x)) = x^j f(x)$.

In particular it contains elements like $X^2D + D^3X + I$ etc.

This ring is clearly non-commutative, but, unlike the linear transformations corresponding to matrices e.g. $M_n(F)$, the ring W is a domain, which is not the case for $M_n(F)$ since, for example

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

i.e. $M_n(F)$ has zero-divisors.

Partitions of a Positive Integer

For a positive integer n, a partition of n is a sum of positive integers that adds up to n.

$n = 1$

$$1 = 1$$

(no other possibilities)
$n = 2$

$$\begin{aligned} 2 &= 2 \\ &= 1 + 1 \end{aligned}$$

$n = 3$

$$\begin{aligned} 3 &= 3 \\ &= 2 + 1 \\ &= 1 + 1 + 1 \end{aligned}$$

Ok, not too difficult, but as n increases, the number of possibilities increases.

Consider $n = 5$

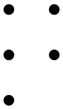
$$
\begin{aligned}
5 &= 5 \\
&= 4 + 1 \\
&= 3 + 2 \\
&= 3 + 1 + 1 \\
&= 2 + 2 + 1 \\
&= 2 + 1 + 1 + 1 \\
&= 1 + 1 + 1 + 1 + 1
\end{aligned}
$$

A way to understand these partitions, and to help see the 'symmetries' that exist amongst them is through what are known as Ferrers Diagrams.

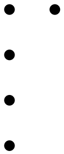
We've been writing these partitions following a certain convention.

Namely, we write partitions typically as $n = i_1 + i_2 + \cdots + i_k$ where $i_1 \geq i_2 \geq \cdots \geq i_k$, such as $5 = 2 + 2 + 1$.

We can represent this as a Ferrers Diagram which is a 'stack' of dots arranged in rows of non-increasing size.



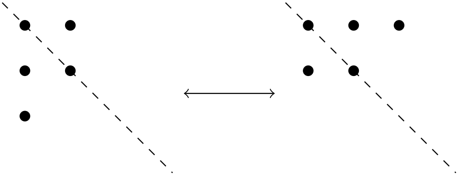
$5 = 2 + 2 + 1$



$5 = 2 + 1 + 1 + 1$

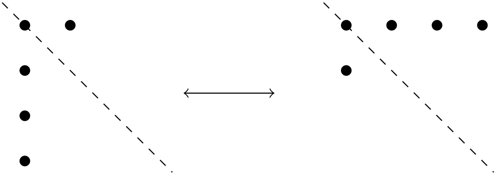
$5 = 1 + 1 + 1 + 1 + 1$

What is rather nice about these diagrams is that they can be 'flipped' to yield different partitions.



$5 = 2 + 2 + 1$ $\qquad\qquad$ $5 = 3 + 2$

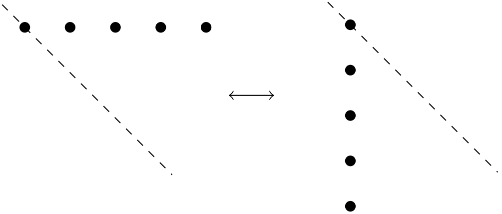
So we say these partitions are 'conjugate', and similarly



$5 = 2 + 1 + 1 + 1$ $\qquad\qquad$ $5 = 4 + 1$

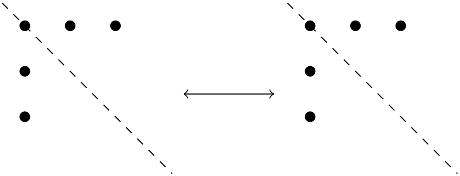
are conjugate as well.

Indeed all partitions can grouped in 'pairs' of partitions which are conjugates of each other, even these two:



$5 = 5$ $\qquad\qquad$ $5 = 1 + 1 + 1 + 1 + 1$

but some can be 'self conjugate', for example:



$5 = 3 + 1 + 1$ $\qquad\qquad$ $5 = 3 + 1 + 1$

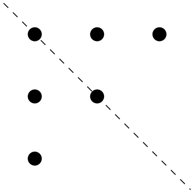
where the 'self conjugacy' is due to the symmetry of the diagram when flipped.

Applied to the $n = 5$ we have the following correspondences:

$$\begin{aligned} 5 &= 5 \\ &= 4+1 \\ &= 3+2 \\ &= 3+1+1 \\ &= 2+2+1 \\ &= 2+1+1+1 \\ &= 1+1+1+1+1 \end{aligned}$$

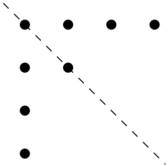
Applied to the $n = 6$ we have these correspondences:

$$\begin{aligned}
6 &= 6\\
&= 5+1\\
&= 4+2\\
&= 4+1+1\\
&= 3+3\\
&= 3+2+1\\
&= 3+1+1+1\\
&= 2+2+2\\
&= 2+2+1+1\\
&= 2+1+1+1+1\\
&= 1+1+1+1+1+1
\end{aligned}$$

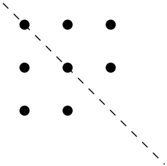


One may wonder if there are ever more than one self-conjugate partition of an integer.

Yes, consider $n = 8$.



$8 = 4 + 2 + 1 + 1$



$8 = 3 + 3 + 2$

In general, the number of self-conjugate partitions is a non-decreasing function of n.

And, speaking of numbers, how many partitions are there of an arbitrary integer n?

Definition

For $n \geq 0$ an integer, let $\mathcal{P}(n)$ be the number of partitions of n.

Consider:

- $\mathcal{P}(0) = 1$(i.e. $0 = 0$ only)
- $\mathcal{P}(1) = 1$
- $\mathcal{P}(2) = 2$
- $\mathcal{P}(3) = 3$
- $\mathcal{P}(4) = 5$
- $\mathcal{P}(5) = 7$
- $\mathcal{P}(6) = 11$

So the question is whether there is an obvious pattern, and also is $\mathcal{P}(n) \approx n$ as we see in these initial values?

We can give some indication of the answer to the second question by looking at $\mathcal{P}(n)$ for larger n.
$\mathcal{P}(10) = 42$

$$
\begin{aligned}
10 &= 10\\
10 &= 9+1\\
10 &= 8+2\\
10 &= 8+1+1\\
10 &= 7+3\\
10 &= 7+2+1\\
10 &= 7+1+1+1\\
10 &= 6+4\\
10 &= 6+3+1\\
10 &= 6+2+2\\
10 &= 6+2+1+1\\
10 &= 6+1+1+1+1\\
10 &= 5+5\\
10 &= 5+4+1\\
10 &= 5+3+2\\
10 &= 5+3+1+1\\
10 &= 5+2+2+1\\
10 &= 5+2+1+1+1\\
10 &= 5+1+1+1+1+1\\
10 &= 4+4+2\\
10 &= 4+4+1+1
\end{aligned}
$$

$10 = 4 + 3 + 3$

$10 = 4 + 3 + 2 + 1$

$10 = 4 + 3 + 1 + 1 + 1$

$10 = 4 + 2 + 2 + 2$

$10 = 4 + 2 + 2 + 1 + 1$

$10 = 4 + 2 + 1 + 1 + 1 + 1$

$10 = 4 + 1 + 1 + 1 + 1 + 1 + 1$

$10 = 3 + 3 + 3 + 1$

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