25.1-1

Find the first four terms of the power series for (i) $\dfrac{1+4x}{1+5x+x^2}$.

For this we have

$$\frac{1+4x}{1+5x+x^2} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

which makes sense since the denominator $1+5x+x^2$ is a power series with a non-zero constant term and is therefore invertible.

So let's solve for a_0, a_1, a_2, a_3.

$$\frac{1+4x}{1+5x+x^2} = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

implies that $(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots)(1 + 5x + x^2) = 1 + 4x$

So let's expand the left hand side which yields

$$\begin{aligned}
&(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots)+\\
&(5a_0x + 5a_1x^2 + 5a_2x^3 + 5a_3x^4 + \dots)+\\
&(a_0x^2 + a_1x^3 + a_2x^4 + a_3x^5 + \dots)\\
&= 1 + 4x
\end{aligned}$$

which when we collect terms can be rewritten as follows:

$$a_0 + (a_1 + 5a_0)x + (a_2 + 5a_1 + a_0)x^2 + (a_3 + 5a_2 + a_1)x^3 + \cdots = 1 + 4x$$

which means $a_0 = 1$, and (since $a_1 + 5a_0 = 4$) implies that $a_1 = -1$.

Continuing, we have that $a_2 + 5a_1 + a_0 = 0$ which means $a_2 - 5 + 1 = 0$, i.e. $a_2 = 4$.

Lastly, $a_3 + 5a_2 + a_1 = 0$ implies that $a_3 = -19$.

As such

$$\frac{1 + 4x}{1 + 5x + x^2} = 1 - x + 4x^2 - 19x^3 + \ldots$$

25.1-4

What is the inverse of $1 + x^2$ in $\mathbb{Z}_2[[x]]$?

We know that, in $\mathbb{Z}[[x]]$ one has

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$
$$= \sum_{n=0}^{\infty} x^n$$

and so (also in $\mathbb{Z}[[x]]$)

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$

which means that, in $\mathbb{Z}_2[[x]]$, since $-1 = 1$ it must be the case that

$$\frac{1}{1+x^2} = 1 + x^2 + x^4 + x^6 + x^8 + \dots$$

25.2-1

Use the 'cover-up' rule to find the partial fraction decomposition of (i) $\dfrac{3+4x}{(1-x)(2+x)}$.

The book's term 'cover-up' is basically the utilization of the fact we saw that, since the denominators are distinct linear factors, they are zero for two different values.

As such when we write

$$\frac{3+4x}{(1-x)(2+x)} = \frac{A}{1-x} + \frac{B}{2+x}$$

$$\downarrow$$

$$3+4x = B(1-x) + A(2+x)$$

we can solve for A and B by 'plugging in' $x = 1$ and $x = -2$.

That is:

$$3 + 4x = B(1 - x) + A(2 + x)$$

implies that $7 = A(3)$ and $-5 = B(3)$ so that $A = \frac{7}{3}$ and $B = -\frac{5}{3}$.

25.2-5

Find the partial fractions decomposition *over the complex field* $\mathbb{C}$ of $(1-x^4)^{-1}$.

We have that $(1-x^4)=(1-x^2)(1+x^2)=(1-x)(1+x)(1+x^2)$ over $\mathbb{R}$ since $1+x^2$ has no real roots.

However, over $\mathbb{C}$ we have $(1-x^4)=(1-x)(1+x)(x+i)(x-i)$ which yields this decomposition:

$$\frac{1}{1-x^4}=\frac{A}{1-x}+\frac{B}{1+x}+\frac{C}{x+i}+\frac{D}{x-i}$$

where we observe that the four roots in question $\{1,-1,i,-i\}$ are distinct.

So for

$$\frac{1}{1-x^4} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{C}{x+i} + \frac{D}{x-i}$$

we have

$$1 = A(1+x)(x+i)(x-i) + B(1-x)(x+i)(x-i) + \\ C(1-x)(1+x)(x-i) + D(1-x)(1+x)(x+i)$$

and so, letting $x = 1$ yields $1 = 4A$, letting $x = -1$ yields $1 = 4B$, so $A = \frac{1}{4}$ and $B = \frac{1}{4}$.

Next, letting $x = i$ yields
$1 = D(1-i)(1+i)(2i) = D(1-i^2)(2i) = D(2)(2i) = D(4i)$ so $D = \frac{1}{4i}$ and similarly if we let $x = -i$ we get $1 = C(1+i)(1-i)(-2i) = C(-4i)$ so $C = -\frac{1}{4i}$.

25.4-4

Let $A(x)$ be the generating function for the sequence (a_n) and define $s_n = a_0 + a_1 + \cdots + a_n$ for each $n \geq 0$.
Show that the generating function for (s_n) is $S(x) = \frac{A(x)}{1-x}$.

Since $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$ then we can show that

$$\begin{aligned}
xA(x) &= a_0x + a_1x^2 + a_2x^3 + a_3x^4 + \dots \\
x^2A(x) &= a_0x^2 + a_1x^3 + a_2x^4 + a_3x^5 + \dots \\
x^3A(x) &= a_0x^3 + a_1x^4 + a_2x^5 + a_3x^6 + \dots \\
&\vdots
\end{aligned}$$

which means that $A(x) + xA(x) + x^2A(x) + x^3A(x) + \dots$ can be represented as a series, namely:

$$(a_0) + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + (a_0 + a_1 + a_2 + a_3)x^3 + \dots$$

where now the coefficients are precisely the s_n defined above.

And the point is that

$$A(x) + xA(x) + x^2A(x) + x^3A(x) + \dots = A(x)(1 + x + x^2 + \dots)$$

which is precisely $A(x)\frac{1}{1-x} = \frac{A(x)}{1-x}$.

25.5-3

Suppose that the sequence (z_n) is defined by

$$z_0 = 1 \quad z_{n+1} = \frac{z_n - a}{z_n - b}$$

for $n \geq 0$ where a and b are real numbers where $b \neq 1$.

Show that if the sequence (u_n) satisfies $\frac{u_{n+1}}{u_n} = z_n - b$ then

$$u_{n+2} - (b-1)u_{n+1} + (a-b)u_n = 0$$

for $n \geq 0$.

I've yet to figure this one out!

25.6-3

Show that the generating function for the sequence defined by

$$u_0 = 1 \; u_{n+1} - 2u_n = na^n$$

is

$$U(x) = \frac{1}{1-2x} + \frac{ax^2}{(1-2x)(1-ax)^2}$$

provided $a \neq 2$. Hence find a formula for u_n.
What happens when $a = 2$?

We first rewrite $u_{n+1} - 2u_n = na^n$ as $u_{n+1} = 2u_n + na^n$ and consider the series:

$$U(x) = u_0 + u_1 x + u_2 x^2 + u_3 x^3 + \dots$$

and apply the recurrence to get

$$\begin{aligned}
U(x) &= u_0 + (2u_0 + 0a^0)x + (2u_1 + 1a^1)x^2 + (2u_2 + 2a^2)x^3 + \dots \\
&= u_0 + (2u_0 x + 2u_1 x^2 + 2u_2 x^3 + \dots) + (0a^0 x + 1a^1 x^2 + 2a^2 x^3 + \dots) \\
&= u_0 + (2u_0 x + 2u_1 x^2 + 2u_2 x^3 + \dots) + (1a^1 x^2 + 2a^2 x^3 + \dots) \\
&= u_0 + 2x(u_0 + u_1 x + u_2 x^2 + \dots) + x^2(1a^1 + 2a^2 x + 3a^3 x^2 + \dots) \\
&= u_0 + 2xU(x) + x^2\left(\frac{d}{dx}\frac{1}{1-ax}\right) \\
&= 1 + 2xU(x) + x^2\left(\frac{a}{(1-ax)^2}\right)
\end{aligned}$$

which can be rearranged as follows:

$$U(x)(1-2x) = 1 + x^2\left(\frac{a}{(1-ax)^2}\right)$$

And this can be rearranged still further to yield the equation asserted in the statement of the problem:

$$U(x) = \frac{1}{1-2x} + \left(\frac{ax^2}{(1-2x)(1-ax)^2}\right)$$

where the right hand side can now be de-composed using partial fractions

$$\frac{ax^2}{(1-2x)(1-ax)^2} = \frac{A}{1-2x} + \frac{B}{1-ax} + \frac{C}{(1-ax)^2}$$

$$\downarrow$$

$$\begin{aligned}
ax^2 &= A(1-ax)^2 + B(1-2x)(1-ax) + C(1-2x) \\
&= A(1-2ax+a^2x^2) + B(1-(a+2)x+2ax^2) + C(1-2x) \\
&= A - 2aAx + Aa^2x^2 + B - B(a+2)x + 2aBx^2 + C - 2Cx \\
&= (A+B+C) + (-2aA-(a+2)B-2C)x + (a^2A+2aB)x^2
\end{aligned}$$

which implies (by comparing coefficients) that

$$\begin{aligned}
A + B + C &= 0 \\
-2aA - (a+2)B - 2C &= 0 \\
a^2A + 2aB &= a
\end{aligned}$$

So let's solve this system.
Since $A + B + C = 0$ we can set $C = -A - B$ so that $-2aA - (a+2)B - 2C = 0$ is equivalent to

$$(2 - 2a)A - aB = 0$$

and in conjunction with the third equation $a^2A + 2aB = 0$ we find that $A = \frac{a}{4-4a+a^2}$ so the denominator cannot be zero or else this is undefined.

And $4 - 4a + a^2 = a^2 - 4a + 4 = (a-2)^2$ so $a \neq 2$ is necessary to solve for A.

In a similar fashion we find that

$$B = \frac{2-2a}{4-4a+a^2}$$

and so

$$C = \frac{1}{a-2}$$

again reinforcing why $a \neq 2$ is important.

Thus

$$U(x) = \frac{1}{1-2x} + \frac{a}{(a-2)^2}\frac{1}{1-2x} + \frac{2-2a}{(a-2)^2}\frac{1}{1-ax} + \frac{1}{a-2}\frac{1}{(1-ax)^2}$$

So what about a formula for u_n?

We know that $\frac{1}{1-ax} = \sum_{n=0}^{\infty} a^n x^n$ which (if we take derivatives) implies

$$\frac{a}{(1-ax)^2} = \sum_{n=1}^{\infty} n a^n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a^{n+1} x^n$$
$$\downarrow$$
$$\frac{1}{(1-ax)^2} = \sum_{n=0}^{\infty} (n+1) a^n x^n$$

and since

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} 2^n x^n$$
$$\frac{1}{1-ax} = \sum_{n=0}^{\infty} a^n x^n$$

then we have that

$$u_n = 2^n + \frac{2^n a}{(a-2)^2} + \frac{a^n(2-2a)}{(a-2)^2} + \frac{(n+1)a^n}{a-2}$$

So the remaining question to ask is, what happens when $a = 2$?

Well, actually the problem can still be solved, it's just that the equation

$$U(x) = \frac{1}{1-2x} + \left(\frac{ax^2}{(1-2x)(1-ax)^2}\right)$$

simplifies to

$$\begin{aligned} U(x) &= \frac{1}{1-2x} + \left(\frac{2x^2}{(1-2x)^3}\right) \\ &= \frac{1-4x+6x^2}{(1-2x)^3} \\ &= 1 + 6x + 24x^2 + 80x^3 + 240x^4 + \dots \end{aligned}$$

What's kind of interesting is that while

$$u_n = 2^n + \frac{2^n a}{(a-2)^2} + \frac{a^n(2-2a)}{(a-2)^2} + \frac{(n+1)a^n}{a-2}$$

is not well defined at $a = 2$, it does have a limit as $a \to 2$ and indeed

$$\lim_{a \to 2} u_n = -\frac{1}{4}2^n n + \frac{1}{4}2^n n^2 + 2^n$$

and for $n = 0$ we get 1, and for $n = 1$ we get 6, $n = 2$ we get 24 which are exactly the coefficients of the series for $U(x)$ above, i.e.

$$U(x) = \frac{1 - 4x + 6x^2}{(1-2x)^3} = 1 + 6x + 24x^2 + 80x^3 + 240x^4 + \dots$$