MA294 - Turn in 5 - Answers

August 11, 2021

(1) Define the sequence $\{a_n\}$ as follows:

$$a_0 = 1, a_1 = 1, a_{n+2} = 2a_{n+1} + 3a_n + 2^n \text{ for } n \geq 2$$

and for this sequence, use a generating function to give an exact formula for a_n in terms of n.

If $A(x)$ is the generating function for this sequence then we can write

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

where, $a_0 = 1$ and $a_1 = 1$ and for $n \geq 2$ the recurrence is used.

As such, we have

$$
\begin{aligned}
a_2 &= 2a_1 + 3a_0 + 2^0 \\
a_3 &= 2a_2 + 3a_1 + 2^1 \\
a_4 &= 2a_3 + 3a_2 + 2^2 \\
&\vdots
\end{aligned}
$$

So

$$
\begin{aligned}
A(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots \\
&= a_0 + a_1x + (2a_1 + 3a_0 + 2^0)x^2 + (2a_2 + 3a_1 + 2^1)x^3 + (2a_3 + 3a_2 + 2^2)x^4 + \dots \\
&= a_0 + a_1x + (2a_1 + x^2 + 2a_2x^3 + 2a_3x^4 + \dots)+ \\
&(3a_0x^2 + 3a_1x^3 + 3a_2x^4 + \dots)+ \\
&(2^0x^2 + 2^1x^3 + 2^2x^4 +)\dots \\
&= a_0 + a_1x + 2x(a_1x + a_2x^2 + \dots)+ \\
&3x^2(a_0 + a_1x + \dots)+ \\
&x^2(2^0 + 2^1x + 2^2x^2 + \dots) \\
&= x + 2xA(x) + 3x^2A(x) + x^2(\frac{1}{1-2x})
\end{aligned}
$$

So

$$A(x)(1-2x-3x^2) = x + \frac{x^2}{1-2x}$$

where $(1-2x-3x^2) = (1-3x)(1+x)$ which means

$$A(x) = \frac{x}{(1-3x)(1+x)} + \frac{x^2}{(1-3x)(1+x)(1-2x)}$$

which, after partial fractions decomposition yields

$$A(x) = -\frac{1}{6}\frac{1}{1+x} + \frac{1}{2}\frac{1}{1-3x} - \frac{1}{3}\frac{1}{1-2x}$$

Since

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$
$$\frac{1}{1-3x} = 1 + 3x + 3^2x^2 + \dots$$
$$\frac{1}{1-2x} = 1 + 2x + 2^2x^2 + \dots$$

we have finally that

$$a_n = -\frac{1}{6}(-1)^n + \frac{1}{2}(3^n) - \frac{1}{3}(2^n)$$