Exercises 25.1

1 Find the first four terms in the power series for

$$\text{(i)}\ \frac{1+4x}{1+5x+x^2}, \qquad \text{(ii)}\ \frac{2+6x+x^2}{3+x+5x^2+x^3}.$$

(You may assume that the coefficients belong to the field $\mathbb{R}$ of real numbers.)

2 Show that the inverse of $1+x$ in $\mathbb{R}[[x]]$ is

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots,$$

where the coefficient of x^n is $(-1)^n$.

3 Using long division, show that the inverse of $1+x+x^2$ in $\mathbb{Z}_2[[x]]$ is

$$(1+x+x^2)^{-1} = 1 + x + x^3 + x^4 + x^6 + x^7 + \cdots.$$

Guess the rule for the coefficients, and prove it.

4 What is the inverse of $1+x^2$ in $\mathbb{Z}_2[[x]]$? What is its inverse in $\mathbb{Z}_3[[x]]$?

25.2 Partial fractions

In this section we begin the task of finding an alternative way to compute the coefficients in the power series for $a(x)/b(x)$, when $a(x)$ and $b(x)$ are polynomials and $b_0 \neq 0$. Although the algorithmic method of long division is adequate for many purposes, it does not yield a formula for the coefficients, and there are circumstances in which such a formula is desirable.

The key to the problem is the decomposition of $a(x)/b(x)$ into *partial fractions*, a technique which may be familiar to some readers. Roughly speaking, if there is a factorization $b(x) = s(x)t(x)$, where $s(x)$ and $t(x)$ have no non-trivial common factor, then we can write

$$\frac{a(x)}{b(x)} = \frac{f(x)}{s(x)} + \frac{g(x)}{t(x)}$$

for some polynomials $f(x)$ and $g(x)$. For example, the polynomial $2-3x+x^2$ is equal to $(1-x)(2-x)$, and we have the equation

$$\frac{5-3x}{2-3x+x^2} = \frac{2}{1-x} + \frac{1}{2-x}.$$

It must be stressed that all the 'fractions' occurring in such equations are actually power series, although in practice it is both convenient and harmless to retain the fractional notation. Note also that it is only necessary to discuss the case when the degree of $a(x)$ is strictly less than the degree of $b(x)$. This is because we have

$$a(x) = b(x)q(x) + r(x)$$

where either $r(x)$ is the zero polynomial or $\deg r(x) < \deg b(x)$, and consequentl

$$\frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)}.$$

Thus the problem is reduced to finding the partial fractions for $r(x)/b(x)$. F example, The cover-up rule is very useful, but there are two points to remember. First, it only gives the numerator of the highest power of a linear factor, so the other coefficients must be obtained in some other way. Secondly, it should only be used when the coefficients are presumed to be real or complex numbers: this is because the technique of substituting values depends upon the assumption that a polynomial and the corresponding polynomial function are interchangeable, and this is false when the coefficients belong to a finite field (Section 22.8). Conventionally, we assume that we are working in the field $\mathbb{C}$ of complex numbers unless some other field is stated explicitly. One good reason for this convention is that any polynomial can be decomposed into linear factors in $\mathbb{C}[x]$, and so the complete decomposition of a 'quotient' into partial fractions with constant numerators can always be achieved.

Exercises 25.2

1 Use the cover-up rule to find the partial fraction decomposition of

(i) $\dfrac{3+4x}{(1-x)(2+x)}$; (ii) $\dfrac{2+x+x^2}{(1+x)(2+x)(3+x)}$.

2 Find the partial fraction decompositions of

(i) $\dfrac{1+3x}{1-3x^2+2x^3}$, (ii) $\dfrac{-5+3x}{6-11x+6x^2-x^3}$.

3 Find the partial fraction decomposition over $\mathbb{Z}_3$ of $1/(1+x^4)$. (Use the factorization obtained in Section 22.8.)

4 Let $h(x)$ be a polynomial with $\deg h(x) < m$. Show that if $q_i(x)$ and γ_i $(1 \le i \le m)$ are defined by

$$h(x) = (\alpha - x)q_1(x) + \gamma_m,$$
$$q_{i-1}(x) = (\alpha - x)q_i(x) + \gamma_{m-i+1} \quad (2 \le i \le m),$$

then

$$h(x) = \gamma_m + \gamma_{m-1}(\alpha - x) + \cdots + \gamma_1(\alpha - x)^{m-1}.$$

5 Find the partial fraction decomposition *over the complex field* $\mathbb{C}$ of $(1-x^4)^{-1}$.

25.3 The binomial theorem for negative exponents

The binomial theorem obtained in Section 11.3 can be formulated as a result of multiplication in the ring $\mathbb{Z}[x]$ of polynomials with integer coefficients. Specifically, it asserts that the coefficient of x^n in the polynomial $(1+x)^k$ is the binomial number $\binom{k}{n}$, that is

$$(1+x)^k = \binom{k}{0} + \binom{k}{1}x + \binom{k}{2}x^2 + \cdots + \binom{k}{n}x^n + \cdots + \binom{k}{k}x^k.$$

In this chapter we have come to regard $(1+x)^{-1}$ as a power series with integer coefficients

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots,$$

and so in the ring of power series $\mathbb{Z}[[x]]$ we can define $(1+x)^{-m}$ to be the produc of m factors $(1+x)^{-1}$. Clearly this definition works for any integer $m \ge 1$, an naturally we would like to have a formula for the coefficient of x^n in the resultin power series. It turns out that the formula is remarkably simple.

Thus, given the extended definition of $\binom{k}{n}$ for positive *and* negative integers k, we have shown that the coefficient of x^n in $(1+x)^k$ is $\binom{k}{n}$. If we include the trivial case $(1+x)^0 = 1$, we have the following general form of the binomial theorem, which holds for all integers k:

$$(1+x)^k = \binom{k}{0} + \binom{k}{1}x + \binom{k}{2}x^2 + \cdots + \binom{k}{n}x^n + \cdots.$$

The general form is a power series, but when k is a positive integer we have

$$\binom{k}{n} = 0 \quad \text{whenever} \quad n > k,$$

so the right-hand side reduces to a polynomial.

There are several trivial extensions of the binomial theorem: formulae for $(a+x)^k$, $(a-x)^k$, and so on. In particular, we shall make considerable use of the formula for $(1-ax)^{-m}$, which is

$$(1-ax)^{-m} = 1 + max + \cdots + \binom{m+n-1}{n}a^n x^n + \cdots.$$

The binomial theorem has been presented here as a rule for finding the coefficients in the power series $(1+x)^k$. Many other power series can be transformed by algebraic manipulations into a form involving the basic power series, and so the binomial theorem can be used to find formulae for their coefficients. For example, in Section 25.1 we obtained the power series

$$\frac{1+3x}{1-2x+x^2} = 1 + 5x + 9x^2 + 13x^3 + \cdots.$$

Noting that the denominator is $(1-x)^2$ we can work out a formula for the general coefficient as follows.

$$\begin{aligned}\frac{1+3x}{1-2x+x^2} &= (1+3x)(1-x)^{-2}\\ &= (1+3x)(1+2x+\cdots+(n+1)x^n+\cdots).\end{aligned}$$

The coefficient of x^n in the product is $(n+1)+3n$, which gives the answer $4n+1$ as one might guess from the first few values.

In later sections of this chapter we shall see as to how the technique of partial fractions enables us to reduce many power series to forms in which the binomial theorem can be applied.

Exercises 25.3

1 Write down, and simplify wherever possible, the coefficient of

(i) x^3 in $(1+2x)^7$, (ii) x^n in $(1-x)^{-4}$,

(iii) x^{2r} in $(1-x)^{-r}$.

2 Write down the first four terms and the general term in the power series $(1-x)^{-3}$.

3 Let a_n be the coefficient of x^n in the power series $(1-x-x^2)^{-1}$. Show that

$$a_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{n-r}{r},$$

where r is the greatest integer such that $0 \le r \le n-r$.

4 What is the coefficient of x^n in the power series

$$\frac{1+2x+2x^2}{1-3x+3x^2-x^3}?$$

5 If $a_0, a_1, \ldots, a_{6r}$ and $b_0, b_1, \ldots, b_{3r}$ are defined by

$$(1-x+x^2)^{3r} = a_0 + a_1x + a_2x^2 + \cdots + a_{6r}x^{6r},$$
$$(1+x)^{3r} = b_0 + b_1x + b_2x^2 + \cdots + b_{3r}x^{3r},$$

show that

$$\sum_{i=0}^{3r} a_i b_{3r-i} = \binom{3r}{r}.$$

25.4 Generating functions

The solution of a combinatorial problem can often be expressed as a sequence (u_n). In such cases, it is often appropriate to use methods based on the representation of (u_n) as a power series

$$U(x) = u_0 + u_1x + u_2x^2 + \cdots.$$

In this context $U(x)$ is traditionally referred to as the **generating function** for the sequence (u_n). (Strictly speaking it should be referred to as the *ordinary* generating function, since mathematicians often use other kinds of generating function. But in this book the ordinary generating function is the only kind needed, so we shall omit the word 'ordinary'.) Of course, we have gone to some lengths to emphasize that $U(x)$ is *not* a function; it is simply an alternative way of writing the sequence (u_n) so that certain algebraic manipulations can be carried out.

Perhaps the best example of a generating function arises from the binomial theorem. The formula

$$(1+x)^k = \binom{k}{0} + \binom{k}{1}x + \binom{k}{2}x^2 + \cdots + \binom{k}{n}x^n + \cdots,$$

can be regarded as saying that the generating function for the sequence defined by

$$u_n = \binom{k}{n},$$

for any given integer k, is

$$U(x) = (1+x)^k.$$

In the rest of this chapter we shall study the generating functions of sequences defined by recursion equations. The method we shall employ is in three stages

(i) use the recursion for (u_n) to obtain an equation for $U(x)$;
(ii) solve the equation for $U(x)$;
(iii) use algebra (specifically, the partial fraction decomposition and the binomial theorem) to find a formula for the coefficients of $U(x)$.

The following *Example* gives a typical application of the method. The sequence considered was also discussed in Section 19.2.

so that

$$u_n = 3^n - 2^n.$$

A simple calculation shows that Mr Stackenboom was a millionaire at the end of thirteen years. □

Exercises 25.4

1 Use the generating function method to find a formula for u_n when the sequence (u_n) is defined by

$$u_0 = 1, \quad u_1 = 1, \quad u_{n+2} - 4u_{n+1} + 4u_n = 0 \quad (n \geq 0).$$

2 Suppose $A(x)$ is the generating function for the sequence (a_n). What are the generating functions for the sequences (p_n), (q_n), and (r_n) defined as follows?

(i) $p_n = 5a_n$; (ii) $q_n = a_n + 5$; (iii) $r_n = a_{n+5}$.

3 Show that $x(1+x)/(1-x)^3$ is the generating function for the sequence whose nth term is n^2.

4 Let $A(x)$ be the generating function for the sequence (a_n) and define

$$s_n = a_0 + a_1 + \cdots + a_n \quad (n \geq 0).$$

Show that the generating function for (s_n) is

$$S(x) = \frac{A(x)}{1-x}.$$

Use this result in conjunction with Ex. 3 to find a formula for $\sum_{i=0}^{n} i^2$.

25.5 The homogeneous linear recursion

In Section 19.2 we found the explicit solution of the recursion

$$u_0 = c_0, \quad u_1 = c_1, \quad u_{n+2} + a_1 u_{n+1} + a_2 u_n = 0 \quad (n \geq 0).$$

This is the case $k = 2$ of the **homogeneous linear recursion** defined by the equations

$$\text{[HLR]} \qquad \begin{cases} u_0 = c_0, \quad u_1 = c_1, \quad \ldots \quad , \quad u_{k-1} = c_{k-1}, \\ u_{n+k} + a_1 u_{n+k-1} + \cdots + a_k u_n = 0 \quad (n \geq 0). \end{cases}$$

We shall now use the method of generating functions to derive the general solution to [HLR] for any value of k.

Theorem 25.5.1 The generating function for the sequence (u_n) defined by [HLR] is

$$U(x) = \frac{R(x)}{1 + a_1 x + \cdots + a_k x^k},$$

where $R(x)$ is a polynomial and $\deg R(x) < k$.

Proof Consider the product

$$\begin{aligned}(1 + a_1 x + \cdots + a_k x^k)U(x) \\ &= (1 + a_1 x + \cdots + a_k x^k)(u_0 + u_1 x + \cdots + u_n x^n + \cdots).\end{aligned}$$

Using the multiplication rule, the coefficient of x^{n+k} is

$$u_{n+k} + a_1 u_{n+k-1} + \cdots + a_k u_n \quad (n \geq 0).$$