Hence u_n is given by a formula

$$u_n = (An^2 + Bn + C)2^n + D(-1)^n,$$

where A, B, C, D are constants. Substituting the given values of u_0, u_1, u_2, u_3 we obtain the equations

$$\begin{aligned} C + D &= 0, \\ 2A + 2B + 2C - D &= -9, \\ 16A + 8B + 4C + D &= -1, \\ 72A + 24B + 8C - D &= 21. \end{aligned}$$

The standard methods of solving linear equations yield the solution $A = 1$, $B = -1$, $C = -3$, $D = 3$. Hence

$$u_n = (n^2 - n - 3)2^n + 3(-1)^n. \qquad \square$$

Exercises 25.5

1 Use the auxiliary equation method to find a formula for u_n when the sequence (u_n) is defined by

(i) $u_0 = 1,\quad u_1 = 3,\quad u_{n+2} - 3u_{n+1} - 4u_n = 0 \quad (n \geq 0)$;

(ii) $u_0 = 2,\quad u_1 = 0,\quad u_2 = -2,$
$u_{n+3} - 6u_{n+2} + 11u_{n+1} - 6u_n = 0 \quad (n \geq 0)$;

(iii) $u_0 = 1,\quad u_1 = 0,\quad u_2 = 0,$
$u_{n+3} - 3u_{n+1} + 2u_n = 0 \quad (n \geq 0)$.

2 Professor McBrain climbs stairs in an erratic fashion. Sometimes he takes two stairs in one stride, sometimes only one. Find a formula for b_n, the number of different ways in which he can climb n stairs.

3 Suppose the sequence (z_n) is defined by

$$z_0 = 1, \qquad z_{n+1} = \frac{z_n - a}{z_n - b} \quad (n \geq 0),$$

where a and b are real numbers and $b \neq 1$. Show that if the sequence (u_n) satisfies

$$u_{n+1}/u_n = z_n - b,$$

then

$$u_{n+2} + (b-1)u_{n+1} + (a-b)u_n = 0 \quad (n \geq 0).$$

Hence find a formula for z_n when $a = 0$ and $b = 2$.

4 Let (u_n), (v_n), (w_n) be the sequences defined by $u_0 = v_0 = w_0 = 1$ and

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \\ w_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_n \\ v_n \\ w_n \end{bmatrix} \quad (n \geq 0).$$

Show that (u_n) satisfies a homogeneous linear recursion.

25.6 Non-homogeneous linear recursions

In some cases the method of generating functions can be used to solve the *non-homogeneous* linear recursion

$$u_0 = c_0, \quad u_1 = c_1, \quad \ldots, \quad u_{k-1} = c_{k-1},$$
$$u_{n+k} + a_1 u_{n+k-1} + \cdots + a_k u_n = f(n) \quad (n \geq 0).$$

The applicability of the method depends upon the particular form of $f(n)$. Roughly speaking, the technique is to work out

$$(1 + a_1 x + a_2 x^2 + \cdots + a_k x^k)U(x)$$

and hope that the terms involving $f(n)$ can be dealt with by some special device.

which yield the solution $A = -\frac{2}{9}$, $B = \frac{1}{4}$, $C = \frac{5}{36}$, $D = -\frac{1}{6}$. On rearranging, we obtain the formula

$$u_n = \tfrac{1}{36}[(-2)^{n+3} + 3^{n+2} - 6n - 1].$$

□

There are many other types of problems which can be dealt with by the generating function method. It is particularly powerful when combined with analytic methods of dealing with power series, but that is beyond the scope of this book. Fortunately, some very important results can be obtained by the purely algebraic means already at our disposal, and several results of this kind will be discussed in the next two chapters.

Exercises 25.6

1 Show that the generating function for the sequence (u_n) defined by the recursion

$$u_0 = 1, \qquad u_{n+1} - 2u_n = 4^n \quad (n \geq 0)$$

is

$$U(x) = \frac{1-3x}{(1-2x)(1-4x)}.$$

Deduce that $u_n = 2^{2n-1} + 2^{n-1}$.

2 Let q_n be the number of words of length n in the alphabet $\{a, b, c, d\}$ which contain an odd number of b's. Show that

$$q_{n+1} = 4^n + 2q_n \quad (n \geq 1).$$

[Hint: partition the set of such words of length $n+1$ into those which begin with b and those which do not.] Hence find the generating function $Q(x)$ for (q_n), assuming $q_0 = 0$, and show that

$$q_n = \tfrac{1}{2}(4^n - 2^n).$$

3 Show that the generating function for the sequence defined by

$$u_0 = 1, \qquad u_{n+1} - 2u_n = n\alpha^n \quad (n \geq 0)$$

is

$$U(x) = \frac{1}{1-2x} + \frac{\alpha x^2}{(1-2x)(1-\alpha x)^2},$$

provided $\alpha \neq 2$. Hence find a formula for u_n. What happens when $\alpha = 2$?

25.7 Miscellaneous Exercises

1 Which of the following are invertible in $\mathbb{R}[[x]]$, and which of them are invertible in $\mathbb{Z}[[x]]$?

(i) $1 + x$; (ii) x^2; (iii) $3 + 2x^3$.

2 Write down the first four terms and the general term in the power series for $(1+x)^{-5}$.

3 Find the coefficient of x^n in the power series for

$$\frac{26 - 60x + 25x^2}{(1-2x)(1-5x)^2}.$$

4 Find the first four terms and the coefficient of x^n in the power series for

$$\frac{1 - x - x^2}{(1-2x)(1-x)^2}.$$

5 Find a formula for the coefficient of x^n in the power series for $(1+x)/(1+x+x^3)$ in $\mathbb{Z}_2[[x]]$.

6 Find the partial fraction decomposition over $\mathbb{Z}_5$ of

$$\frac{4x+2}{x^3 + 2x^2 + 4x + 3}.$$

7 Use the method of Ex. 25.4.4 to find formulae for $\sum i^3$ and $\sum i^4$, where the sums are taken over the range $1 \leq i \leq n$.

8 If $A(x)$ is the generating function for the sequence (a_n), define the *derivative* $A'(x)$ to be the generating function for the sequence (a'_n) where $a'_n = (n+1)a_{n+1}$ $(n \geq 0)$. Show that

$$(AB)'(x) = A'(x)B(x) + A(x)B'(x).$$

9 By taking the derivative of the binomial expansion show that

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = 2^{n-1}n.$$

10 Use the formula $(1-x^2)^{-n} = (1-x)^{-n}(1+x)^{-n}$ to establish the identity

$$\sum_{i=0}^{r}(-1)^i\binom{n+r-i-1}{r-i}\binom{n+i-1}{i} = \begin{cases} 0 & \text{if } r \text{ is odd} \\ \binom{n+\frac{1}{2}r-1}{\frac{1}{2}r} & \text{if } r \text{ is even.} \end{cases}$$

11 Find an expression for $(1-x)^{-n}(1-x^k)^n$ as a polynomial of degree $n(k-1)$, and hence derive an identity

$$\binom{r}{n-1}\binom{n}{0} - \binom{r-k}{n-1}\binom{n}{1} + \binom{r-2k}{n-1}\binom{n}{2} - \cdots = 0,$$

valid when $r \geq nk$.

12 Write down the generating function for the sequence (u_n) whose terms represent the number of ways of distributing n different books to four people.

13 Let c_r be the number of ways in which the total r can be obtained when four dice are thrown. Show that the generating function for the sequence (c_r) is

$$C(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^4.$$

14 Write down the generating function for the numbers b_r of integers n in the range $0 \leq n \leq 10^m - 1$ for which the sum of the digits in the base 10 representation is r.

15 Use the generating function method to solve the following recursion:

$$u_0 = 2, \quad u_1 = -6, \quad u_{n+2} + 8u_{n+1} - 9u_n = 8(3^{n+1}) \quad (n \geq 0).$$

16 Find the general form of the solution to the recursion

$$y_{n+2} - 6y_{n+1} + 9y_n = 2^n + n \quad (n \geq 0).$$

17 When $k \geq 0$ let $f(n, k)$ be the number of k-subsets of $\{1, 2, \ldots, n\}$ which do not contain two successive integers. Show that

$$f(n, k) = f(n-2, k-1) + f(n-1, k).$$

Let $F_k(x)$ be the generating function of the numbers $f(n, k)$ for fixed k. Find a recursion for $F_k(x)$ and deduce that

$$f(n, k) = \binom{n-k+1}{k}.$$

18 Use the generating function method to solve the recursion

$$u_0 = 1, \qquad u_{n+1} = 3u_n + 2^{n-1} \quad (n \geq 0).$$

19 The *exponential* generating function for the sequence (u_n) is defined to be the power series

$$\tilde{u}(x) = u_0 + \frac{u_1}{1!}x + \frac{u_2}{2!}x^2 + \cdots + \frac{u_i}{i!}x^i + \cdots.$$

Use the recursion $d_n = nd_{n-1} + (-1)^n$ for the derangement numbers to show that the exponential generating function for (d_n) is $e^{-x}/(1-x)$. Hence derive the usual explicit formula for d_n.

20 Let $\tilde{Q}(x)$ be the exponential generating function for the numbers q_n, where q_n is the number of partitions of an n-set. Use the formula obtained in Ex. 12.7.10 to prove that

$$\tilde{Q}(x) = \exp(e^x - 1).$$