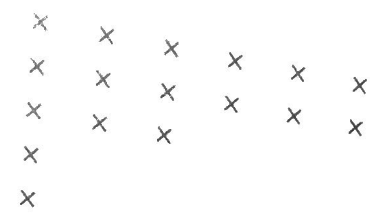

represent the partitions

$$5+3+2 \qquad\qquad\qquad 6+6+3+1+1.$$

(Such a diagram is often called a *Ferrers diagram*, or a *Ferrers graph*, after its inventor N. M. Ferrers. His name is frequently misspelled.)

This simple idea is very useful for proving theorems about partitions. A good example is the following result.

Theorem 26.1 Let n and r be any positive integers. The number of partitions of n in which there are not more than r parts is equal to the number of partitions of $n+r$ in which there are exactly r parts.

Proof Suppose we are given a partition of $n+r$ with exactly r parts. Its diagram has exactly r marks in the first column, and if we delete this column we obtain the diagram of a partition of n into at most r parts. Conversely, if a partition of the second kind is given, we can add a new first column of r marks and obtain a partition of the first kind. In other words, there is a bijection between the sets of partitions of the two kinds, and so they have the same cardinality. (The bijection is illustrated in Fig. 26.1.) $\square$

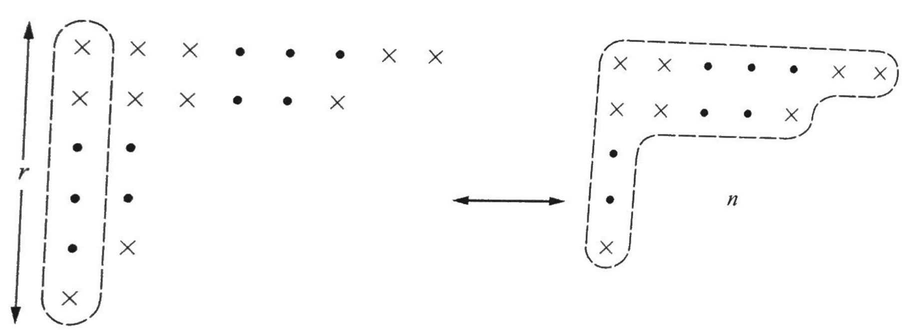

Fig. 26.1 Illustrating the proof of Theorem 26.1.

If $\mathcal{P}$ is a property of partitions we shall use the notation $p(n \mid \mathcal{P})$ to denote the number of partitions of n which satisfy the property $\mathcal{P}$. With this notation, Theorem 26.1 can be expressed in the form

$$p(n \mid \text{number of parts } \leq r) = p(n+r \mid \text{number of parts} = r).$$

Exercises 26.1

1 Write down the diagrams representing the following partitions:

(i) $[1^2 3^3 5\, 7]$, (ii) $[2\, 4\, 6^2 7]$.

2 Find the values of $p(n)$, $1 \leq n \leq 7$, by listing all the possibilities.

3 Let

$$p_k(n) = p(n \mid \text{number of parts} = k).$$

Show that the following formula (obtained in Ex. 12.4.2) is a consequence of Theorem 26.1:

$$p_k(n) = p_k(n-k) + p_{k-1}(n-k) + \cdots + p_1(n-k),$$

and use it to calculate $p(8)$.

4 Use an argument based on diagrams to prove that the number of partitions of n which have at most m parts is equal to the number of partitions of $n + \frac{1}{2}m(m+1)$ in which there are m parts, all of them different. [Hint: add a 'triangle' with $\frac{1}{2}m(m+1)$ marks.]

26.2 Conjugate partitions

The diagrammatic representation of partitions suggests the following simple method of transforming one partition into another: switch the rows and columns of the diagram. For example, the partition $\lambda = [1^2 2 3^2 4]$ is transformed into the partition $\lambda' = [1\,3\,4\,6]$, as the diagram indicates.

$$\begin{array}{cccc} \times & \times & \times & \times \\ \times & \times & \times & \\ \times & \times & \times & \\ \times & \times & & \\ \times & & & \\ \times & & & \end{array} \quad \to \quad \begin{array}{cccccc} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & & \\ \times & \times & \times & & & \\ \times & & & & & \end{array}$$

In general, partitions λ and λ' related in this way are said to be **conjugate**.

The transformation $\lambda \to \lambda'$ which takes a partition to its conjugate can be used to prove some useful results. A typical example is as follows. If the largest part of λ is m, then the first row of the diagram for λ and the first column of the diagram for λ' both contain m marks. Thus λ' is partition with m parts. In other words, the transformation $\lambda \to \lambda'$ is a bijection from the set of partitions of n with the largest part equal to m to the set of partitions of n with exactly m parts. So we have proved that

$$p(n \mid \text{largest part} = m) = p(n \mid \text{number of parts} = m).$$

A partition λ is said to be **self-conjugate** if $\lambda = \lambda'$.

Theorem 26.2 The number of self-conjugate partitions of n is equal to the number of partitions of n whose parts are distinct and odd.

Proof The first row and first column of the diagram for a self-conjugate partition of n contain the same number of marks, say k. Since they have one common mark, the total number of marks in the first row and column is the odd number $2k - 1$. In the same way, when the first row and column are removed, the remainder of the second row and column contain an odd number of marks, say $2l - 1$. Continuing, we obtain a partition of n in which each part is odd, and the parts are all distinct. The procedure may be visualized as a 'straightening' of the L-shaped sections of the given self-conjugate partition (Fig. 26.2).

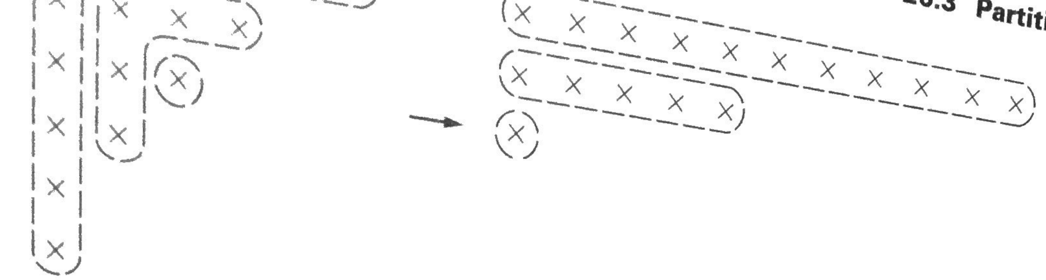

Fig. 26.2
Illustrating the proof of Theorem 26.2.

Conversely, if we are given a partition of n with parts which are odd and distinct, then we can 'fold up' the diagram to obtain a self-conjugate partition of n. Hence the numbers of partitions of the two kinds are equal. □

Exercises 26.2

1 Write down the conjugates of the following partitions in standard notation:

(i) $[1^2356]$, (ii) $[2^23^358]$.

2 Which of the following partitions are self-conjugate?

(i) $[1^224]$, (ii) $[1^3456]$,
(iii) $[2^235^2]$, (iv) $[1^42348]$.

3 Use the method described in Theorem 26.2 to list the self-conjugate partitions of 20.

4 Show that the number of self-conjugate partitions of n with largest part equal to k is the same as the number of self-conjugate partitions of $n-2k+1$ with the largest part not exceeding $k-1$. Hence find the number of self-conjugate partitions of 41 in which the largest part is 11.

26.3 Partitions and generating functions

In this section we shall obtain a formula for the generating function

$$P(x) = p(0) + p(1)x + p(2)x^2 + \cdots$$

where $p(n)$ is the number of partitions of n and, *by convention*, $p(0) = 1$.

The starting point is the formula

$$(1 - x^i)^{-1} = 1 + x^i + x^{2i} + x^{3i} + \cdots,$$

which should be regarded as an equation for the inverse of $1 - x^i$ in the ring of power series $\mathbb{C}[\![x]\!]$. The formula can be obtained by direct calculation, or by replacing x by x^i $(i \geq 1)$ in the standard formula for $(1-x)^{-1}$. For our purposes, the key observation is that this power series is the generating function for the numbers f_n of partitions of a very simple kind

$$f_n = p(n \mid \text{each part is equal to } i).$$

For clearly there are no such partitions unless n is a multiple of i, when there is just one partition $n = i + i + \cdots + i$. In other words

$$f_n = \begin{cases} 1 & \text{if } n = \alpha i \quad (\alpha = 0, 1, 2, \ldots), \\ 0 & \text{otherwise.} \end{cases}$$