power series $(1-x^i)^{-1}$ in such a way that the product of the terms is x^n. In other words, the generating function for the numbers $p^{(m)}(n)$ is

$$P^{(m)}(x) = (1-x)^{-1}(1-x^2)^{-1}\cdots(1-x^m)^{-1}.$$

For any given value of n we have $p(n) = p^{(n)}(n)$, since a partition of n cannot contain any parts greater than n. This corresponds to the fact that the terms

$$(1-x^i)^{-1} = 1 + x^i + x^{2i} + \cdots$$

with $i > n$ do not contribute to the coefficient of x^n in the infinite product $P(x)$. In other words, the coefficient of x^n in $P(x)$ is the same as the coefficient of x^n in $P^{(n)}(x)$, and this is just $p^{(n)}(n) = p(n)$. Thus $P(x)$ is the generating function for the sequence $(p(n))$. □

As indicated in the last paragraph of the proof, the appearance of an infinite number of factors in the product $P(x)$ is not a real difficulty, because in order to calculate any given coefficient it is necessary to consider only a finite number of factors. We shall encounter some other infinite products of power series in the following sections, and they will all share this property. For this reason, we shall make no further reference to the supposed difficulties of dealing with expressions of this kind.

Exercises 26.3

1 Write down the generating functions for the sequences whose nth terms are

(i) the number of partitions of n into parts equal to 3 or 5;

(ii) the number of partitions of n into parts equal to 2, 4, or 6.

2 By multiplying the relevant power series, determine the coefficient of x^9 in

$$\frac{1}{(1-x)(1-x^2)(1-x^3)}.$$

Interpret your result as the number of partitions of a certain kind, and check your answer by listing the partitions explicitly.

3 In how many ways can one pound (100 p) be exchanged for coins of the values 50p, 20p, 10p, and 5p?

4 In how many ways can a total weight of 15 kg be made up from weights of 1 kg, 2 kg, and 4 kg?

26.4 Generating functions for restricted partitions

Simple modifications of the arguments given in Section 26.3 enable us to wri down generating functions for the numbers of partitions satisfying various restri tions. For instance, suppose we stipulate that each part must occur no more thar times. Then the contributions from the parts equal to i correspond to the terms the polynomial

$$1 + x^i + x^{2i} + \cdots + x^{ki},$$

rather than the power series $(1-x^i)^{-1} = 1 + x^i + x^{2i} + \cdots$, and so the genera function is

$$(1 + x + \cdots + x^k)(1 + x^2 + \cdots + x^{2k})(1 + x^3 + \cdots + x^{3k})\cdots.$$