Exercises 26.4

1 Write down formulae for the generating functions for the sequences whose nth terms are

(i) $p(n \mid \text{each part occurs at most twice})$,
(ii) $p(n \mid \text{each part is power of 2})$,
(iii) $p(n \mid \text{the smallest part is 5})$.

2 Use the method of generating functions to find the number of partitions of 16 in which each part is an odd prime.

3 Write down the generating function for the sequence whose nth term is

$$p(n \mid \text{no even number occurs more than once as a part}).$$

Hence show that this number is equal to

$$p(n \mid \text{each part occurs at most 3 times}).$$

[Hint: $(1-y^4) = (1-y)(1+y+y^2+y^3)$.]

4 Show that

$$(1-x)(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^m}) = 1-x^{2^{m+1}},$$

and derive the formula

$$(1-x)^{-1} = (1+x)(1+x^2)(1+x^4)\cdots,$$

where the right-hand side is the product of the factors $1+x^{2^r}$ for all $r \geq 0$. Deduce that every positive integer has a unique partition whose parts are distinct powers of 2.

26.5 A mysterious identity

We have shown that the generating function for the numbers of partitions with distinct parts is

$$D(x) = (1+x)(1+x^2)(1+x^3)\cdots.$$

For example, the partition $7 = 1+2+4$ corresponds to the term $x \times x^2 \times x^4$ and contributes 1 to the coefficient of x^7. Now consider as to what happens when we change all the + signs to −, so that we obtain the power series

$$\begin{aligned} Q(x) &= (1-x)(1-x^2)(1-x^3)\cdots \\ &= 1 + q_1 x + q_2 x^2 + \cdots. \end{aligned}$$

Here again each partition of n with distinct parts makes a contribution to the coefficient of x^n, but now the contribution is $(-1)^d$, where d is the number of parts. For example, the partition $7 = 1+2+4$ corresponds to the term

$$(-x) \times (-x^2) \times (-x^4) = (-1)^3 x^7$$

and contributes -1 to x^7. In general, each partition of n with an even number of distinct parts contributes 1 to q_n, and each partition with an odd number of distinct parts contributes -1. Thus

$$q_n = e_n - o_n,$$

where

$$\begin{aligned} e_n &= p(n \mid \text{ the parts are distinct and even}), \\ o_n &= p(n \mid \text{ the parts are distinct and odd}). \end{aligned}$$

Exercises 26.5

1 Show that the coefficient of x^n in the power series expansion of

$$(1+x)(1+x^2)(1+x^3)\cdots$$

is even unless n is a number of the form $\frac{1}{2}m(3m \pm 1)$.

2 Let $q_{n,\,d}$ denote the number of partitions of n which have d parts, all distinct. Show that

$$q_{n,\,d} = q_{n-d,\,d} + q_{n-d,\,d-1} \quad (1 \le d \le n).$$

3 Use the recursion equation of Ex. 2 to calculate $q_{n,\,d}$ for $1 \le n \le 22$ and $1 \le d \le 6$, and verify that your answers are compatible with Ex. 1.

4 Show that the mysterious identity can also be written in the form

$$\prod_{i=1}^{\infty}(1-x^i) = \sum_{m=-\infty}^{\infty}(-1)^m x^{\frac{1}{2}m(3m-1)}.$$

26.6 The calculation of $p(n)$

It is possible to calculate $p(n)$ by simply listing all the partitions of n in some order, but this is clearly a very inefficient procedure. We can do better by using a recursive approach, such as the one given in Ex. 26.1.3, but that particular method is also rather inefficient. A very good recursive procedure is based upon the mysterious identity established in Section 26.5, and this is the method which we shall study now.

The alert reader will have spotted that the generating function $P(x)$ for the numbers $p(n)$ is the inverse of the generating function $Q(x)$ for $q_n = e_n - o_n$; specifically,

$$P(x) = (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\cdots,$$
$$Q(x) = (1-x)(1-x^2)(1-x^3)\cdots.$$

It follows that

$$\begin{aligned} P(x)Q(x) &= (1 + p(1)x + p(2)x^2 + \cdots + p(n)x^n + \cdots) \\ &\quad \times (1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots) \\ &= 1. \end{aligned}$$

Thus the coefficient of x^n $(n \ge 1)$ in the product is zero. By the rule for multiplying power series, we obtain the equation

$$p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - p(n-15) + \cdots = 0.$$

We can rewrite this in the form

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \cdots,$$

where there are finitely many terms on the right-hand side, since only those terms $p(n-k)$ with $n-k \ge 0$ appear. For example,

$$p(13) = p(12) + p(11) - p(8) - p(6) + p(1).$$

These formulae provide an efficient recursive method for computing $p(n)$. The calculation may be set out as in Table 26.6.1, where we assume for convenience that the values $p(0) = 1$, $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, $p(5) = 7$, $p(6) = 11$ are already known.

Table 26.6.1

n	7	8	9	10	11	12	13	14
$p(n-1)$	11	15	22	30	42	56	77	101
$p(n-2)$	7	11	15	22	30	42	56	77
$p(n-5)$	2	3	5	7	11	15	22	30
$p(n-7)$	1	1	2	3	5	7	11	15
$p(n-12)$	—	—	—	—	—	1	1	2
$p(n)$	15	22	30	42	56	77	101	135

Of course, extra rows of the table are required for the calculation of further values of $p(n)$.

So we have achieved our stated aim of finding an efficient method for the calculation of $p(n)$—just how efficient it is will be established in the following exercises. Remarkably, although the final form of the method is so simple, almost all the theory covered in this chapter is necessary to justify it.

Exercises 26.6

1 Extend the table as far as $p(20)$.

2 Find the least value of n for which $p(n) > 1000$.

3 Let $t(n)$ denote the number of terms on the right-hand side of the equation for $p(n)$. Show that

$$t(n) = \begin{cases} 2m-1 & \text{if } \tfrac{1}{2}m(3m-1) \le n < \tfrac{1}{2}m(3m+1), \\ 2m & \text{if } \tfrac{1}{2}m(3m+1) \le n < \tfrac{1}{2}(m+1) \\ & \quad \times(3m+2). \end{cases}$$

Deduce that there is a constant K such that $t(n) \le K\sqrt{n}$ for all $n \ge 1$.

4 Show that the number of additions and subtractions required to compute $p(n)$ by the method of this section is $O(n^{3/2})$.

26.7 Miscellaneous Exercises

1 Calculate $p_5(9)$ and $p_5(10)$, where $p_k(n)$ is the number of partitions of n into k parts.

2 Use the Ferrers diagram to prove that the number of partitions of n in which each part is 1 or 2 is equal to the number of partitions of $n+3$ which have exactly two distinct parts.

3 Show that the number of partitions of n with at most two parts is $\lfloor \frac{1}{2}n \rfloor$.

4 Write down the generating functions for the number of

(i) partitions of n in which each part is at most 5;

(ii) partitions of n in which only even parts can occur more than once;

(iii) partitions of n in which each part is a multiple of 3.

5 Give an alternative description of $p(n) - p(n-1)$ as the number of partitions of n which have a certain property.

6 Show that

$$p(n+2) - 2p(n+1) + p(n) \ge 0.$$

7 Show that the number of partitions of $2n$ into three parts, such that the sum of any two parts is greater than the third, is equal to the number of partitions of n with exactly three parts.

8 Show that the number of partitions of n into k parts satisfies

$$p_k(n) \ge \frac{1}{k!}\binom{n-1}{k-1}.$$

9 Verify that if λ is a partition with parts $\lambda_1, \lambda_2, \ldots, \lambda_r$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ then the parts of the conjugate partition λ' are given by

$$\lambda'_i = \text{the number of parts of } \lambda \text{ not less than } i \ (1 \leq i \leq \lambda_1).$$

10 Suppose that λ and μ are partitions of n with parts $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_s$, respectively. Show that λ and μ are conjugate if and only if

$$\lambda_1 = s, \quad \mu_1 = r, \quad \lambda_i + \mu_j \neq i + j - 1 \quad (1 \leq i \leq r, 1 \leq j \leq s).$$

11 Show that the number of partitions of $n - m$ into exactly $k - 1$ parts, none of the parts being greater than m, is equal to the number of partitions of $n - k$ into $m - 1$ parts, none of the parts being greater than k.

12 Show that the number of partitions of n in which no part occurs more than $2^{k+1} - 1$ times is equal to the number of partitions of n in which no multiple of 2^k can occur more than once as a part.

13 The following rule is the basis for a method of listing all partitions of n in **lexicographic order**. The first partition is $[n]$. Suppose that the current partition λ has parts $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. Then the next partition is found as follows:

(i) if $\lambda_r \neq 1$, then the parts of the next partition are $\lambda_1, \lambda_2, \ldots, \lambda_{r-1}, \lambda_r - 1, 1$;

(ii) if $\lambda_r = \lambda_{r-1} = \cdots = \lambda_{r-s+1} = 1$ but $\lambda_{r-s} = x \neq 1$, then the parts of the next partition are obtained by replacing $\lambda_{r-s}, \lambda_{r-s+1}, \ldots, \lambda_r$ by $x - 1, x - 1, x - 1, \ldots, x - 1, y$, where $1 \leq y \leq x - 1$ and the number of parts $x - 1$ is chosen so that the result is a partition of n.

Use this rule to list the partitions of 8.

14 Write a program based on the algorithm outlined in the previous exercise.

15 Write a program to calculate $p(n)$ for $n \leq 100$ by the method of Section 26.6.

16 In a Ferrers diagram the largest possible square of marks in the top left-hand corner is called a **Durfee square**. Show that for each self-conjugate partition of n there is an associated integer k and a partition of n into one part of size k^2 (the Durfee square) and some parts of even size not greater than $2k$. Deduce that

$$(1 + x)(1 + x^3)(1 + x^5)\cdots = 1 + \sum_{k=1}^{\infty} \frac{x^{k^2}}{(1 - x^2)(1 - x^4)\cdots(1 - x^{2k})}.$$

17 Use the Durfee square construction to show that the partition generating function $P(x)$ satisfies the identity

$$P(x) = 1 + \sum_{k=1}^{\infty} \frac{x^{k^2}}{(1 - x)^2(1 - x^2)^2\cdots(1 - x^k)^2}.$$

18 Show that

$$(1 + x^2)(1 + x^4)(1 + x^6)\cdots = 1 + \sum_{k=1}^{\infty} \frac{x^{k(k+1)}}{(1 - x^2)(1 - x^4)\cdots(1 - x^{2k})}.$$